Physical Realizability of Bordism and Homotopy Classes by Gapped Lattice Systems

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Abstract

The invertible condensed phase spectrum $\mathbf{IP}^{\mathrm{cond}}_{d,G}$ of Part IV assigns to each short-range-entangled phase an abstract class in a bordism- or homotopy-theoretic classification. The present paper asks the converse question: which of those classes are actually produced by an explicit uniformly gapped lattice Hamiltonian? We formalize realizability as the essential surjectivity, on π_0, of the comparison map ρ from the stabilized stack of lattice models to the abstract spectrum, and we separate two nested notions: realizability by *some* gapped local Hamiltonian, and realizability by a local *commuting-projector* model. On the positive side we record, in the condensed language, three facts that are theorems today: every group-cohomology class in $H^{d+1}(G, U(1))$ is realized by an explicit G-symmetric commuting-projector model carrying the prescribed edge anomaly (Chen–Gu–Liu–Wen; Else–Nayak); in spatial dimension one Ogata's operator-algebraic index is a *complete* invariant of symmetry-protected phases with on-site finite symmetry, so the realized set coincides with the invariant set there, while in dimension two the same index is known to exist and every in-cohomology class is realized but its completeness is open; and commuting-projector realizations face a hard obstruction: no local commuting-projector Hamiltonian carries a nonzero electric Hall conductance (Kapustin–Fidkowski, under $U(1)$) or a nonzero chiral central charge (Kapustin–Spodyneiko), so every chiral invertible class lies outside the commuting-projector image. We assemble the known constructions (group-cohomology models, Walker–Wang models for beyond-cohomology classes in $3+1$ dimensions, the E_8 state, the Kitaev Majorana chain, cluster states) and the known obstructions into a dimension-by-symmetry status table separating the torsion part from the free part. We then state three numbered conjectures on the realizability image: that it is exactly the short-range-entangled subspectrum; that chiral classes are realizable by non-commuting gapped Hamiltonians though never by commuting projectors; and that realizability descends over profinite disorder hulls $\Omega = Q^{\mathbb{Z}^d}$ precisely when a condensed-cohomological obstruction vanishes. Accompanying Haskell computes the Kitaev-chain $\mathbb{Z}_2$ invariant across the parameter plane and the cluster-state string order, verifying quantization, phase-boundary detection, and degradation under a symmetry-breaking perturbation.

Contents

1 Introduction

1.1 The realizability gap

A classification theorem for topological phases produces an abstract invariant: a bordism group, a stable-homotopy group, a value of some generalized cohomology theory. Part IV of this series constructs the target of such a classification inside the condensed-mathematics program: the invertible condensed phase spectrum $\mathbf{IP}^{\mathrm{cond}}_{d,G}$, a connective condensed spectrum whose homotopy groups are meant to enumerate the invertible gapped phases in spatial dimension d with symmetry G. What a spectrum enumerates and what a laboratory, or a finite-dimensional local Hilbert space, can produce are not the same thing. A class $c \in \pi_0\mathbf{IP}^{\mathrm{cond}}_{d,G}$ is an equivalence class of abstract data; to say that c is *physically realizable* is to exhibit a concrete uniformly gapped local Hamiltonian whose phase maps to c. The prospectus for this program names the discrepancy plainly: the gap between abstract homotopy-TQFT classifications and explicit local lattice realizations is a recognized issue, and not every abstract invertible field theory is known to arise from a microscopic gapped system.

This paper is about that gap. We take the classification of Part IV as given and study the image of the comparison map

$$\rho\colon \pi_0\,\mathrm{Shape}\big(\mathfrak{Phase}_{d,G}\big) \longrightarrow \pi_0\mathbf{IP}^{\mathrm{cond}}_{d,G},$$

from the set of phases of genuine lattice systems (components of the stabilized condensed stack of Part III) to the abstract spectrum. Realizability is the question of which classes lie in $\operatorname{im}\rho$; more precisely, since a phase is by definition already a class of lattice systems, realizability is the essential surjectivity of ρ onto a physically-meaningful subspectrum. We will argue that the honest present-day answer has three parts: a large *constructive* region where explicit models are known; a sharp *obstructed* region governed by a no-go theorem; and a residual *open* region.

1.2 Two notions of realizability

The single most important distinction in this paper is between two nested classes of Hamiltonians. Say that a class c is

- *gapped-realizable* if some uniformly gapped local (or quasi-local) Hamiltonian has phase c; and
- *commuting-projector realizable* (CP*-realizable*) if some Hamiltonian $H = \sum_x P_x$ with each P_x a finite-range projector, $[P_x, P_y] = 0$, has phase c.

Commuting-projector models are the exactly-solvable backbone of the field: they have exact ground states, exact gaps, string operators, and a transparent entanglement structure. Every CP-realizable class is gapped-realizable, but (and this is the crux) the converse fails, and it fails for a reason that is a rigorous theorem rather than a modeling inconvenience. The Kapustin–Fidkowski theorem [18] states that a local commuting-projector Hamiltonian has vanishing Hall conductance. A Chern insulator, an integer quantum Hall state, the E_8 state (all manifestly gapped-realizable, all with nonzero Hall response) are therefore *not* CP-realizable. Conflating the two notions is the characteristic error this subject invites, and much of what follows is an effort to keep them apart.

1.3 What is proved, what is conjectured

In keeping with the stance of the whole series, we label a statement a Theorem or Proposition only when it follows from cited present-day results, and everything else is a numbered Conjecture. The rigorous content of this paper is:

- **Theorem V-A** (Theorem 3.1): every class in $H^{d+1}(G, U(1))$ is CP-realizable, by the Chen–Gu–Liu–Wen fixed-point construction [7] with the Else–Nayak edge characterization [10].
- **Theorem V-B** (Theorem 5.1): in $d = 1$, for on-site finite symmetry, Ogata's operator-algebraic index [34, 33] is a complete invariant, so realized = invariant there; in $d = 2$ the H^3 index exists [35] and every in-cohomology class is realized, but completeness is open (Theorem 5.2).
- **Theorem V-C** (Theorem 6.1): the commuting-projector no-go, zero electric Hall conductance under $U(1)$ (Kapustin–Fidkowski [18]) and zero chiral central charge (Kapustin–Spodyneiko [21]), exactly bounds the CP-image.
- Two elementary Propositions with complete proofs: the Kitaev-chain winding invariant (Theorem 4.3) and the cluster-state string order (Theorem 3.4), both of which are also checked by the accompanying Haskell.

The conjectural content is three numbered statements about the global shape of $\operatorname{im}\rho$ (Section 8): that it is the short-range-entangled subspectrum (Conjecture V-1); that chiral classes are gapped-realizable but never CP-realizable (Conjecture V-2); and that realizability descends over profinite disorder hulls under a condensed-cohomological obstruction (Conjecture V-3).

1.4 Relation to companion papers

This is Part V of a six-part program that recasts topological phases of matter in the condensed-mathematics paradigm of Clausen–Scholze [37]. The parts compose as a modular hierarchy, each taking the previous as input.

Part I, *Condensed Locality* [28], makes the interaction space a Banach space $\mathcal{B}_F$ and promotes the Heisenberg dynamics to a morphism of condensed objects; it supplies the Lieb–Robinson control that gives meaning to a "uniformly gapped family" over a profinite base. Part II, *Positivity and Condensed State Spaces* [30], attaches the quasi-local C^*-algebra and, through Aoki's solidification bridge [2], a solid-K-theory invariant; the operator-K invariants that appear in our status table (Hall conductance, Chern number) are values of that functor. Part III, *The Uniformly Gapped Substack* [32], cuts out the substack $\mathfrak{Gap}_{d,G}$ whose stabilized π_0 is the set $\mathrm{Phases}_{d,G}$ that our map ρ starts from; the robustness of that π_0 under perturbation is what makes "the phase of a lattice model" well defined. Part IV, *From Lattice Models to Effective Field Theories* [29], group-completes the invertible sector into the target spectrum $\mathbf{IP}^{\mathrm{cond}}_{d,G}$; realizability is precisely the surjectivity question for the comparison map that Part IV constructs. Part VI, the synthesis [31], assembles the five modules into a single program and records the Master Conjecture; our obstruction subgroup is one of the two hard walls (the other being the undecidability of the gap in Part III) that bound the synthesis.

The reader needs from Parts III–IV only the following interface, restated in Section 2: a phase is a component of a stabilized condensed stack, stacking gives it an abelian-group structure, and $\mathbf{IP}^{\mathrm{cond}}_{d,G}$ is the invertible part group-completed. No result of this paper depends on the conjectural global stack structure; we use the module-level theorems of the companions and cite them where needed.

1.5 Outline

Section 2 sets up the comparison map and the two realizability notions inside the condensed program. Sections 3 and 4 survey positive constructions: the group-cohomology models and the cluster state, then beyond-cohomology and topological-order models together with the Kitaev Majorana chain. Section 5 states the low-dimension completeness theorem and is careful about its scope. Section 6 states the Kapustin–Fidkowski obstruction and draws the CP-versus-gapped line. Section 7 gathers everything into a status table. Section 8 states the conjectures on the realizability image, and Section 9 treats disorder-robust realizability over profinite hulls. Section 10 describes the computational checks. Sections 11 and 12 discuss limitations and conclude.

2 The realizability question in the condensed program

2.1 The interface from Parts III and IV

We recall only what we need. Fix a spatial dimension d, a lattice L, on-site Hilbert spaces, an F-function locality class, and a symmetry group G with condensation $\underline{G}$. Part I organizes the admissible G-symmetric interactions into the condensed moduli object $\mathfrak{Ham}_{d,G}$, whose value on a profinite probe S is the set of S-continuous families of interactions. Part III singles out the *uniformly* gapped substack

$$\mathfrak{Gap}_{d,G}(S) \;=\; \bigcup_{\Delta>0} \Big\{ H \in \mathfrak{Ham}_{d,G}(S) : \inf_{s\in S} \operatorname{gap}(H_s) \geq \Delta \Big\},$$

where the word *uniformly* (the infimum over the probe) is essential: a family with $\inf_s \operatorname{gap}(H_s) = 0$ is not a member. Inverting the class $\mathcal{W}$ of gapped adiabatic / finite-depth equivalences [3] and stabilizing by product-state ancillas gives the phase ∞-groupoid $\mathfrak{Phase}_{d,G} = (\mathfrak{Gap}_{d,G}[\mathcal{W}^{-1}])^{\mathrm{st}}$, and the set of phases is

$$\mathrm{Phases}_{d,G} \;=\; \pi_0 \operatorname{Shape}\big(\mathfrak{Phase}_{d,G}\big).$$

Stacking $\boxtimes$ makes $\mathfrak{Phase}_{d,G}$ symmetric monoidal; its invertible objects form a Picard ∞-groupoid, and Part IV group-completes them into the connective condensed spectrum $\mathbf{IP}^{\mathrm{cond}}_{d,G}$. Writing $\mathrm{Phases}^{\times}_{d,G} \subseteq \mathrm{Phases}_{d,G}$ for the invertible (short-range-entangled) phases, group completion is a map

$$\gamma\colon\ \mathrm{Phases}^{\times}_{d,G} \longrightarrow \pi_0\mathbf{IP}^{\mathrm{cond}}_{d,G}.$$

2.2 The comparison map and realizability

Part IV's central comparison is between the microscopic invariant computed on the lattice and the deformation class of the associated effective field theory. Composing the lattice phase with that comparison gives a map to the abstract bordism/homotopy classification; denote its value on π_0 by

$$\rho\colon\ \mathrm{Phases}_{d,G} \longrightarrow \mathcal{C}_{d,G},$$

where $\mathcal{C}_{d,G}$ is the abstract classification group of the relevant EFT: for invertible phases, $\pi_0\mathbf{IP}^{\mathrm{cond}}_{d,G}$ after realification, conjecturally a connective cover of Kubota's Ω-spectrum [27] or the Freed–Hopkins bordism target [12]. The direction of ρ matters. Constructing a lattice model and reading off its abstract class is the "forward" direction and is always available. Realizability is the "backward" question: given $c \in \mathcal{C}_{d,G}$, is there a phase $P \in \mathrm{Phases}_{d,G}$ with $\rho(P) = c$?

Definition 2.1 (Realizability)**.** A class $c \in \mathcal{C}_{d,G}$ is *gapped-realizable* if $c \in \operatorname{im}\rho$, i.e. there is a uniformly gapped G-symmetric local Hamiltonian whose phase P satisfies $\rho(P) = c$. It is CP-*realizable* if such an H can moreover be taken to be a local commuting-projector Hamiltonian $H = \sum_x P_x$, $P_x^2 = P_x = P_x^*$, $[P_x, P_y] = 0$, each P_x of finite range. Write

$$\operatorname{im}\rho \;\supseteq\; \operatorname{im}\rho^{\mathrm{CP}}$$

for the two images; $\operatorname{im}\rho^{\mathrm{CP}} \subseteq \operatorname{im}\rho$ always.

The whole paper is the study of the two subsets $\operatorname{im}\rho^{\mathrm{CP}} \subseteq \operatorname{im}\rho \subseteq \mathcal{C}_{d,G}$. We record the reformulation that ties realizability to the language of the program.

Proposition 2.2 (Realizability as essential surjectivity)**.** *c is gapped-realizable if and only if the comparison functor $\mathfrak{Phase}_{d,G} \to \mathcal{C}_{d,G}$ (viewing $\mathcal{C}_{d,G}$ as a discrete groupoid) hits the object c up to isomorphism; that is, $c \in \operatorname{im}\rho$ iff c is in the essential image on π_0. Equivalently, gapped-realizability of all of $\mathcal{C}_{d,G}$ is the statement that ρ is surjective.*

Proof. Immediate from the definitions: π_0 of the comparison functor is the map ρ of sets, and a set map hits c iff c is in its image, which for a functor into a discrete groupoid is the essential image on objects. There is no content beyond unwinding the notation; the content is entirely in *which* c are hit, addressed in the following sections. □

Remark 2.3 (Why CP is not a technicality). One might hope that CP-realizability and gapped-realizability coincide up to stable equivalence, so that the exactly-solvable models exhaust the phases. They do not: Theorem 6.1 exhibits a permanent obstruction. The correct picture is that $\operatorname{im}\rho^{\mathrm{CP}}$ is a proper subgroup of $\operatorname{im}\rho$ whenever chiral phases exist (i.e. for $d = 2$ without antiunitary protection, and in higher dimensions with the appropriate gravitational response). Keeping the two apart is what lets us state positive results (constructions are usually CP) and the obstruction (which is about CP) without contradiction.

2.3 Short-range entanglement

The invertible phases are the short-range-entangled ones [11]: a phase P is invertible for $\boxtimes$ iff there is P' with $P \boxtimes P'$ trivial (a product state) after stabilization. We write $\mathrm{SRE}_{d,G} \subseteq \mathrm{Phases}_{d,G}$ for this subset; group completion γ restricts to it. Non-invertible topological order (nonzero total quantum dimension, anyons that cannot be condensed away) lies outside SRE and outside the spectrum-level classification altogether; it needs the higher-categorical machinery flagged in Part VI and is not classified by $\mathbf{IP}^{\mathrm{cond}}_{d,G}$. Our realizability question for the *spectrum* is therefore a question about SRE; topologically ordered models such as the toric code enter only as CP-realizations of *trivial* invertible class (they are invertible-trivial but intrinsically ordered), and the chiral topological orders (e.g. Kitaev's non-abelian phase) enter through their invertible *edge* data, the chiral central charge.

3 Positive realizability I: group-cohomology SPT phases

3.1 The Chen–Gu–Liu–Wen construction

The first and largest constructive region is the group-cohomology part of the symmetry-protected topological (SPT) classification. Let G be a finite group acting on-site and unitarily; work in spatial dimension d, so the relevant cocycle degree is $d + 1$.

Theorem 3.1 (Cohomological classes are realized; V-A)**.** *For every finite group G and every class $\omega \in H^{d+1}(G, U(1))$ there is an explicit G-symmetric local commuting-projector Hamiltonian H_ω on a lattice with on-site G-representation, such that:*

(i) H_ω has a unique gapped ground state on any closed d-manifold; the ground state is a renormalization-group fixed-point wavefunction and is short-range-entangled (preparable from a product state by a finite-depth circuit once the on-site G-symmetry constraint is dropped — no G-symmetric finite-depth circuit prepares it, which is exactly what makes the phase a nontrivial SPT);

(ii) the phase of H_ω is trivial as an ungauged state but nontrivial as a G-SPT, and the assignment $\omega \mapsto [H_\omega]$ is a group homomorphism $H^{d+1}(G, U(1)) \to \mathrm{SRE}_{d,G}$ that is injective on the group-cohomology subgroup;

(iii) the boundary of H_ω carries an anomalous G-action whose obstruction to being realized on-site is exactly ω.

In particular every group-cohomology class is CP*-realizable.*

Proof. This is the content of [7], with the edge statement (iii) supplied by [10]; we give the structure of the argument and cite the theorems for the analytic steps. Chen–Gu–Liu–Wen construct, from a representative inhomogeneous $(d+1)$-cocycle $\nu_{d+1}\colon G^{d+1} \to U(1)$, a fixed-point wavefunction on a triangulated d-manifold: to each branched triangulation one assigns an amplitude that is a product of cocycle phases ν_{d+1} over the $(d+1)$-simplices of a filling, and the cocycle condition guarantees that the amplitude is independent of the filling and defines a short-range-correlated state. The parent Hamiltonian is a sum of projectors, one per local patch, each projecting onto the local fixed-point subspace; because the fixed-point conditions on overlapping patches are compatible, the projectors commute, giving a commuting-projector Hamiltonian with the wavefunction as its unique frustration-free ground state. That establishes (i). The map on classes is additive because stacking two decorated wavefunctions multiplies their cocycle phases, i.e. adds the cohomology classes; injectivity on the group-cohomology subgroup is the statement that distinct classes give non-adiabatically-connected models, proved by the edge obstruction of (iii). For (iii), Else–Nayak show that the boundary of such an SPT carries a G-action that cannot be extended to an on-site action of G on the boundary Hilbert space alone; the obstruction to on-site-ness lives in $H^{d+1}(G, U(1))$ and equals ω. This anomalous edge action is what protects the bulk phase and certifies nontriviality. □

Remark 3.2 (Scope of Theorem V-A). Theorem V-A covers exactly the *in-cohomology* subgroup of the SPT classification. It does *not* claim to realize classes outside $H^{d+1}(G, U(1))$: the beyond-group-cohomology classes of Kapustin [17], detected by cobordism invariants rather than group cohomology, require different constructions (Section 4). Nor does on-site finiteness extend for free to continuous G or to spatial symmetry; those need their own constructions and are not asserted here.

3.2 The cluster state as a worked example

The one-dimensional cluster state is the smallest nontrivial instance of Theorem 3.1 and the model our Haskell computes with, so we give it explicitly. Take a chain of qubits with the symmetry $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by $X_{\text{even}} = \prod_{j \text{ even}} X_j$ and $X_{\text{odd}} = \prod_{j \text{ odd}} X_j$. The relevant cohomology is $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) \cong \mathbb{Z}_2$, so there is exactly one nontrivial SPT class, and it is realized by the cluster state.

Definition 3.3 (Cluster state)**.** The cluster state $|C\rangle$ on a chain of n qubits is the unique (up to boundary conditions) common $+1$ eigenstate of the commuting stabilizers

$$K_j \;=\; Z_{j-1}\, X_j\, Z_{j+1}, \qquad 1 < j < n,$$

together with boundary stabilizers. It is the frustration-free ground state of $H_{\text{cl}} = -\sum_j K_j$, a commuting-projector Hamiltonian (after $P_j = (1+K_j)/2$).

The SPT order of $|C\rangle$ is invisible to any local order parameter but visible to a nonlocal string operator. The following is elementary and is verified exactly by the Haskell stabilizer engine.

Proposition 3.4 (Cluster-state string order)**.** *Fix $a < b$ with $b-a$ even, and set*

$$S_{a,b} \;=\; Z_a \left(\prod_{r=0}^{(b-a)/2-1} X_{a+1+2r}\right) Z_b \;=\; Z_a\, X_{a+1} X_{a+3} \cdots X_{b-1}\, Z_b.$$

Then in the cluster state $\langle C|S_{a,b}|C\rangle = +1$ exactly, while every single-site expectation vanishes, $\langle C|Z_j|C\rangle = \langle C|X_j|C\rangle = 0$. Moreover, under the symmetry-breaking on-site rotation $U(\theta) = \prod_j e^{-i\theta Z_j/2}$ (which breaks $X_{\text{even}}, X_{\text{odd}}$),

$$\langle C|U(\theta)^*\, S_{a,b}\, U(\theta)|C\rangle \;=\; (\cos\theta)^m, \qquad m = \tfrac{b-a}{2} = \#\{X \text{ factors in } S_{a,b}\},$$

so the string order degrades monotonically from 1 at $\theta = 0$ to 0 at $\theta = \pi/2$.

Proof. The stabilizer group $\mathcal{S} = \langle K_j\rangle$ is abelian and $|C\rangle$ is its unique $+1$ eigenvector, so for a Pauli operator P one has $\langle C|P|C\rangle = +1$ if $P \in \mathcal{S}$, $= -1$ if $-P \in \mathcal{S}$, and $= 0$ otherwise (because then P anticommutes with some stabilizer g, whence $\langle C|P|C\rangle = \langle C|PggP^{-1}\cdots\rangle$ forces a sign flip and the value is its own negative). The telescoping identity

$$K_{a+1}K_{a+3}\cdots K_{b-1} = \big(Z_a X_{a+1} Z_{a+2}\big)\big(Z_{a+2} X_{a+3} Z_{a+4}\big)\cdots = Z_a\, X_{a+1}X_{a+3}\cdots X_{b-1}\, Z_b,$$

where the internal $Z^2 = 1$ cancellations occur at the shared even-offset sites, shows that this product of alternate stabilizers equals $S_{a,b}$ exactly. Hence $S_{a,b} \in \mathcal{S}$ and $\langle C|S_{a,b}|C\rangle = +1$. The single-site Z_j anticommutes with K_j (through the central X_j), and the single-site X_j anticommutes with $K_{j\pm1}$ (through their Z_j factors); hence each has expectation 0. No product of a single Z or X with a stabilizer lands back in $\mathcal{S}$, so these are genuinely zero. For the rotated expectation, $U(\theta)^* Z_j U(\theta) = Z_j$ and $U(\theta)^* X_j U(\theta) = \cos\theta\, X_j - \sin\theta\, Y_j$, so expanding $U^* S_{a,b} U$ over the m rotated X factors gives 2^m Pauli terms. The all-X term is $(\cos\theta)^m S_{a,b}$ with expectation $(\cos\theta)^m$. Any other term differs from $S_{a,b}$ by replacing at least one X_k with $Y_k = iX_kZ_k$, i.e. equals a phase times $S_{a,b} \cdot \prod_{k\in T} Z_k$ for a nonempty set T of the X-sites; since the only X-free element of $\mathcal{S}$ is the identity, $\prod_{k\in T} Z_k \notin \mathcal{S}$ for $T \neq \varnothing$, so $S_{a,b}\prod_{k\in T} Z_k \notin \mathcal{S}$ (up to sign) and its expectation is 0. Thus all cross terms vanish and the total is $(\cos\theta)^m$. Monotone decrease on $[0, \pi/2]$ is clear. □

Remark 3.5. Theorem 3.4 is exactly the SPT-diagnostic behaviour: a nonlocal string order that is quantized in the protected phase and destroyed only when the protecting symmetry is broken, with no accompanying local order parameter. It realizes the nontrivial class of $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) \cong \mathbb{Z}_2$, the smallest case of Theorem 3.1. The Haskell of Section 10 computes $\langle S_{a,b}\rangle$ and its θ-degradation from the stabilizer group directly, with no closed-form shortcut.

4 Positive realizability II: beyond cohomology and topological order

4.1 Beyond group cohomology: Walker–Wang models

Not every invertible or topologically ordered phase is captured by group cohomology. In $3+1$ dimensions there are SPT and symmetry-enriched phases whose invariants are cobordism classes outside the image of group cohomology [17]. A general commuting-projector construction for a large class of these is available.

Proposition 4.1 (Walker–Wang realizations)**.** *Given a premodular (braided fusion) category $\mathcal{B}$, the Walker–Wang construction [43] produces a $3+1$-dimensional local commuting-projector Hamiltonian whose ground state on a closed 3-manifold is a fixed-point state, and whose 2-dimensional boundary carries the topological order associated with $\mathcal{B}$. When $\mathcal{B}$ is modular the bulk is invertible (a $3+1$d SPT-like state) with a chiral boundary; when $\mathcal{B}$ is a symmetric fusion category the construction reproduces group-cohomology and certain beyond-cohomology SPTs.*

Proof. Cited: [43] construct the state sum and its commuting-projector parent Hamiltonian from the premodular data, and identify the boundary theory. We use it only as a source of CP-realizations for classes beyond Theorem 3.1; no new argument is supplied. □

Remark 4.2 (The chiral boundary caveat). Walker–Wang models are CP in the $3+1$d bulk, but a modular input forces the $2+1$d boundary to be chiral, and by Theorem 6.1 that boundary cannot itself be a CP model. This is not a contradiction (the bulk is CP, the boundary is not), but it is a clean illustration of why CP-realizability is a property of a specific model in a specific dimension, not a phase-intrinsic and dimension-blind attribute.

4.2 Chiral invertible phases: the E_8 state

The cleanest bosonic example of the obstruction to come is the E_8 state: a $2+1$d bosonic short-range-entangled phase with no symmetry, nontrivial because its edge carries eight chiral bosons with chiral central charge $c_- = 8$ and a thermal Hall response. It generates the free part $\mathbb{Z}$ of the $2+1$d bosonic invertible classification. The E_8 state is gapped-realizable (it is a genuine gapped phase, described by a K-matrix Chern–Simons theory and realizable in coupled-layer and network constructions), but because $c_- \neq 0$ it has a nonzero thermal Hall conductance, and Theorem 6.1(ii) (Kapustin–Spodyneiko [21]) forbids any local commuting-projector realization. This obstruction rests on the chiral central charge, *not* on the electric Kapustin–Fidkowski no-go: the bosonic E_8 state carries no $U(1)$ charge and has $\sigma_H = 0$, so part (i) of Theorem 6.1 does not apply to it and part (ii) is what does the work. It is the bosonic analogue, at the level of gravitational response, of the fermionic integer quantum Hall state [42]; its nonzero chiral central charge is the edge datum that also governs chiral topological order more broadly [23].

4.3 Fermionic $\mathbb{Z}_2$: the Kitaev Majorana chain

The canonical fermionic example is the Kitaev chain [25], the $1+1$d class-D/BDI superconductor whose two phases are distinguished by a $\mathbb{Z}_2$ invariant and whose topological phase binds an unpaired Majorana zero mode at each end. Its $\mathbb{Z}_2$ label is the class-D entry of the periodic table of free-fermion phases [24], whose systematic form is the operator-K-theoretic classification of topological insulators and superconductors [40, 36] together with its bulk-edge correspondence [6, 26]; the closely related Su–Schrieffer–Heeger chain [38] is its chiral-symmetric cousin. It is the fermionic counterpart of the cluster state, and our Haskell computes its invariant across the parameter plane.

Write the chain with spinless fermions c_j, hopping t, chemical potential μ, and p-wave pairing amplitude Δ_p (written with a subscript to keep it visibly distinct from the uniform gap bound $\Delta = \Delta$ of Section 2):

$$H = \sum_j \Big[-t\,(c_j^\dagger c_{j+1} + \mathrm{h.c.}) - \mu\big(c_j^\dagger c_j - \tfrac{1}{2}\big) + \Delta_\mathrm{p}\,(c_j c_{j+1} + \mathrm{h.c.}) \Big].$$

In momentum space the Bogoliubov–de Gennes Bloch Hamiltonian is $h(k) = d_z(k)\,\tau_z + d_y(k)\,\tau_y$ in the Nambu basis, with

$$d_z(k) = -2t\cos k - \mu, \qquad d_y(k) = 2\Delta_\mathrm{p}\sin k, \qquad d_x(k) \equiv 0,$$

where $\tau_{x,y,z}$ are Pauli matrices in particle–hole space and the τ_x component d_x vanishes identically because the pairing is real. The single-particle spectrum is $\pm\sqrt{d_y^2 + d_z^2}$, so the bulk gap closes exactly when $d_y = d_z = 0$, i.e. at $\sin k = 0$ and $\mu = \mp 2t$, i.e. on the discriminant $|\mu| = 2|t|$ (for $\Delta_\mathrm{p} \neq 0$).

Proposition 4.3 (Kitaev-chain $\mathbb{Z}_2$ invariant)**.** *For $\Delta_\mathrm{p} \neq 0$ and $|\mu| \neq 2|t|$ the map $k \mapsto (d_y(k), d_z(k)) \in \mathbb{R}^2 \setminus \{0\}$, $k \in [0, 2\pi)$, has a well-defined winding number $w \in \mathbb{Z}$ about the origin, and*

$$w = \begin{cases} 1, & |\mu| < 2|t| \quad (\textit{topological}), \\ 0, & |\mu| > 2|t| \quad (\textit{trivial}). \end{cases}$$

Equivalently the Majorana number $\mathcal{M} = \mathrm{sgn}\big(d_z(0)\big)\,\mathrm{sgn}\big(d_z(\pi)\big) = \mathrm{sgn}\big((\mu + 2t)(\mu - 2t)\big) = \mathrm{sgn}(\mu^2 - 4t^2)$ equals -1 in the topological phase and $+1$ in the trivial phase. The invariant is quantized on the gapped locus and jumps precisely on $\Sigma = \{|\mu| = 2|t|\}$; the jump is the 1d instance of the relative transition charge $\partial\nu$ of the program.

Proof. The curve $k \mapsto (d_y(k), d_z(k)) = (2\Delta_\mathrm{p}\sin k,\ -2t\cos k - \mu)$ is an ellipse with semi-axes $2|\Delta_\mathrm{p}|$ (horizontal) and $2|t|$ (vertical) centered at $(0, -\mu)$. For $\Delta_\mathrm{p}, t \neq 0$ it is a nondegenerate closed curve; its winding number about the origin is 1 if the origin lies inside the ellipse and 0 if outside (the ellipse is convex and traversed once). The origin is inside iff $\mu^2/(2t)^2 < 1$, i.e. $|\mu| < 2|t|$, giving the stated w. The Majorana number is the product of the signs of d_z at the two particle–hole-invariant momenta $k = 0, \pi$ (where $d_y = 0$), a standard $\mathbb{Z}_2$ Pfaffian invariant for class D; here $d_z(0) = -2t - \mu$ and $d_z(\pi) = 2t - \mu$, so $\mathcal{M} = \mathrm{sgn}(-2t - \mu)\,\mathrm{sgn}(2t - \mu) = \mathrm{sgn}(\mu^2 - 4t^2)$. Both w mod 2 and $\mathcal{M}$ change value exactly across $|\mu| = 2|t|$, where the curve passes through the origin and the bulk gap closes; on the gapped locus each is locally constant and integer/sign quantized. □

Remark 4.4 (What this realizes, and what it does not). The Kitaev chain CP-realizes nothing chiral: it is $1+1$d and its invariant is torsion ($\mathbb{Z}_2$), and at the fixed point $\mu = 0$, $t = \Delta_{\mathrm{p}}$ it is itself a commuting-(Majorana-)projector model. It illustrates the torsion, CP-realizable corner of the table. The free ($\mathbb{Z}$) invariants live in even space dimension (Chern number, $d = 2$) and are governed by the obstruction of Section 6; the Kitaev chain does not reach them.

In two dimensions the fermionic story is richer and, importantly, also constructive: the group-supercohomology models of Gu–Wen [14] realize a large class of interacting fermionic SPTs, the discrete-spin-structure construction of Tarantino–Fidkowski [39] gives explicit commuting-projector models for *all* $2+1$d fermionic SPTs with symmetry $G_f = G \times \mathbb{Z}_2^f$ (including all eight members of the $\mathbb{Z}_8$ classification for $G = \mathbb{Z}_2$), and the general construction-and-classification programme of Wang–Gu [44] extends this. These are torsion classes and are CP-realizable; they populate the fermionic rows of the status table (Table 1).

4.4 Homotopy of the space of models

Realizability is not only a π_0 question. The higher homotopy of the space of gapped Hamiltonians records adiabatic pumps (π_1) and higher families (π_n); realizing a class in π_n means building an n-parameter family of gapped models with the prescribed higher invariant; the π_1 generator is the adiabatic charge pump of Thouless [41], whose rigorous lattice avatar is the higher Berry class of Kapustin–Sopenko [20], and the homotopical foundations of such parametrized families are set up by Beaudry et al. [4]; a complementary classification by the homotopy of the space of states of a C^*-algebra appears in [8, 9]. For $2+1$d topological order, Aasen–Wang–Hastings [1] propose explicit generators of π_1, π_2, π_3 of the space of Hamiltonians via automorphisms of topological order and honeycomb/automorphism codes. We stress that these homotopy groups are *conjectural*: [1] give constructions and strong evidence but not a proof that these are the full homotopy groups, and we cite them as such. The moduli-space perspective of Hsin–Wang [16] is the closest ordinary-topology precedent for the object Part IV condenses.

5 Completeness in low dimension

In dimension one, realizability is completely settled for on-site finite symmetry, because the operator-algebraic invariant is a *complete* invariant: the realized set is forced to equal the invariant set. In dimension two the same operator-algebraic index is known to exist and every in-cohomology class is realized, but its completeness is open, so only the surjectivity half survives there. The abstract target is the group-cohomology classification, itself a special case of the generalized-cohomology description of SPT phases [13, 45].

Theorem 5.1 (Completeness in one dimension; V-B)**.** *Let G be a finite group acting on-site on a quantum spin chain, and consider G-symmetric gapped ground-state phases in the operator-algebraic (split/approximately-factorized) sense. Then in $d = 1$ the $H^2(G, U(1))$-valued index is a* complete *invariant of the symmetric phase [34, 33]. Consequently, in $d = 1$ with on-site finite symmetry, the map ρ restricted to these SPT phases is a* bijection *onto*

$H^2(G, U(1))$*: every class is realized (by Theorem 3.1) and no two inequivalent models share an index. Realizability is settled there: realized = invariant.*

Proof. The completeness statement is Ogata's theorem: the $d = 1$ classification of G-symmetric gapped phases by the second-cohomology index is proved in [34, 33], for on-site finite G in the stated operator-algebraic framework. Surjectivity onto $H^2(G, U(1))$ is Theorem 3.1 (the CGLW models realize every class). Injectivity of ρ on these phases is exactly the completeness of the index. Composing, ρ is a bijection in $d = 1$. □

Remark 5.2 (Two dimensions: the index exists, completeness is open; V-B′). In $d = 2$ the situation is genuinely weaker, and it is important not to overstate it. Ogata constructs an $H^3(G, \mathbb{T})$-valued index for G-symmetric gapped phases with on-site finite symmetry [35], and proves it is a *well-defined invariant*; combined with the CGLW construction (Theorem 3.1), the map ρ is therefore *surjective* onto the in-cohomology classes $H^3(G, \mathbb{T})$, so every such class is realized. But whether this index is a *complete* invariant (whether it separates all $d = 2$ SPT phases with on-site finite symmetry) is *not* established: Ogata's theorem gives existence and invariance of the index, not completeness. We therefore record $d = 2$ completeness as an **open problem**, not a theorem, and do *not* assert realized = invariant in $d = 2$; only the surjectivity half (every in-cohomology class is realized) is settled there. This matches the identical treatment in Part IV [29].

Remark 5.3 (Scope: do not extrapolate). Theorem V-B is a completeness statement only for *on-site finite* symmetry in $d = 1$; its $d = 2$ analogue is the open problem of Theorem 5.2 (the index exists but completeness is unproven). It must not be read as a completeness statement for:

- continuous symmetry groups (e.g. $U(1)$, $SU(2)$), where the index theory and the classification differ;
- spatial (crystalline) symmetry, which requires the action-stack $[Y/\underline{G}]$ formalism and different invariants;
- dimensions $d \geq 3$, where no analogous completeness theorem is available and where beyond-cohomology classes (Section 4) appear;
- intrinsic topological order, which is not an SPT and not covered by these indices.

Within its scope the theorem is sharp; outside it, realizability is open or obstructed, and conflating the two is precisely the over-extrapolation the program's editorial stance forbids.

6 The obstruction: the Kapustin–Fidkowski no-go

We now state the hard wall. It is the single result that most sharply bounds realizability, and it is about the commuting-projector class specifically.

Theorem 6.1 (Commuting-projector no-go; V-C)**.** *Let $H = \sum_x P_x$ be a local commuting-projector Hamiltonian on a two-dimensional lattice — each P_x a projector of bounded range, $[P_x, P_y] = 0$ — with a gapped ground state.*

(i) Electric (Kapustin–Fidkowski [18]). *If H is $U(1)$-symmetric, the zero-temperature electric Hall conductance of the ground state vanishes, $\sigma_H = 0$.*

(ii) Thermal (Kapustin–Spodyneiko [21]). *The relative chiral central charge of the ground state vanishes, $c_- = 0$, with* no *symmetry hypothesis.*

Consequently: a $U(1)$-symmetric invertible phase with $\sigma_H \neq 0$ (the integer quantum Hall states and Chern insulators) is not CP*-realizable, by (i); and* any *invertible phase with $c_- \neq 0$ (the E_8 state, and every chiral phase) is not* CP*-realizable, by (ii):*

$$c \in \operatorname{im}\rho,\ \big(\sigma_H(c) \neq 0 \textit{ with } U(1)\big) \textit{ or } \big(c_-(c) \neq 0\big) \quad \Longrightarrow \quad c \notin \operatorname{im}\rho^{\mathrm{CP}}.$$

The two obstructions are logically independent: σ_H and c_- are distinct invariants — the bosonic E_8 state has $\sigma_H = 0$ (no $U(1)$ charge) yet $c_- = 8$ — and part (ii) requires no symmetry, whereas part (i) needs $U(1)$.

Proof. Both parts are cited, not reproved. Part (i) is the theorem of Kapustin–Fidkowski [18]: a $U(1)$-symmetric local commuting-projector Hamiltonian has vanishing zero-temperature electric Hall conductance, taken here as the locally-computable, integer-quantized index of [19]. Part (ii) is the theorem of Kapustin–Spodyneiko [21], who derive a Kubo-type formula for the thermal Hall conductance of a two-dimensional lattice system and prove that the relative chiral central charge of any local commuting-projector Hamiltonian vanishes. The stated obstructions are the contrapositives. □

Remark 6.2 (How absolute is the wall?). The vanishing in Theorem 6.1 is for the *ordinary* on-site $U(1)$ electric response (part (i)) and for the standard relative chiral central charge (part (ii)). Recent work of Hsin–Kobayashi [15] realizes nonzero *generalized* Hall conductivities in local commuting-projector models by using symmetries that are not expressible through on-site charge operators; this does not contradict Theorem 6.1 but shows the no-go is sharp about *which* response it annihilates. The invertible-phase obstruction we use (σ_H for $U(1)$ phases (i), c_- for chiral phases (ii)) is unaffected.

Remark 6.3 (Exactly what is and is not obstructed). The theorem obstructs CP-realizability, not gapped-realizability. Chern insulators exist; the integer quantum Hall effect is a laboratory fact; the E_8 state is a genuine gapped phase. What Theorem 6.1 says is that none of these admits an *exactly-solvable commuting-projector* description: their entanglement cannot be brought to a strict finite-depth fixed point compatible with the nonzero gravitational/electromagnetic response. The obstruction is thus a statement about the *method* (exact solvability) as much as about the phase. This is why the honest realizability picture must always name which class of Hamiltonians is meant.

Corollary 6.4 (The CP-image is a proper subgroup when chiral phases exist)**.** *In any (d, G) for which $\mathcal{C}_{d,G}$ contains a class with nonzero Hall conductance or nonzero chiral central charge (in particular $d = 2$ with no antiunitary protection),* $\operatorname{im}\rho^{\mathrm{CP}} \subsetneq \operatorname{im}\rho$*: the inclusion of Theorem 2.1 is strict.*

Proof. By Theorem 6.1 such a chiral class is not in $\operatorname{im}\rho^{\mathrm{CP}}$; it is in $\operatorname{im}\rho$ because the phase exists as a gapped lattice system (Chern insulator / E_8 construction). Hence the inclusion is strict. □

Remark 6.5 (Relation to the completeness theorem). There is no tension with Theorem 5.1: the completeness theorem is about SPT phases with on-site finite symmetry, whose invariants are torsion group-cohomology classes with zero Hall conductance; the obstruction is about chiral (free-part) invertible phases, which are not SPT in that sense and lie outside the scope of Theorem 5.1. The two results partition the $d = 2$ landscape cleanly: torsion SPT (every in-cohomology class CP-realized by Theorem 3.1, with completeness itself open, Theorem 5.2) versus chiral free part (obstructed for CP, open in general).

7 A status table for realizability

We collect the state of knowledge into Table 1, organized by spatial dimension and symmetry class, separating the torsion part of the classification from the free part, and recording for each the best available construction, the applicable obstruction, and whether realizability is settled, constructive, obstructed-for-CP, or open. The entries are the theorems and constructions cited above; nothing in the table is claimed beyond them.

Two patterns organize the table. First, the *torsion* part of the classification is, wherever a construction is known, realized by commuting-projector models, and in $d = 1, 2$ with on-site finite symmetry this is complete. Second, the *free* (chiral) part is gapped-realizable but never CP-realizable, uniformly, by Theorem 6.1. The next section turns these two patterns into conjectures about the whole image.

8 The realizability image: conjectures

The theorems above describe $\operatorname{im}\rho$ and $\operatorname{im}\rho^{\mathrm{CP}}$ in the regions where constructions and completeness results exist. The global shape of the two images is not a theorem today; it is the conjectural content of this paper. We state three numbered conjectures, in decreasing order of confidence, and phrase them in the condensed language so that they interlock with the rest of the program.

8.1 The image is the short-range-entangled subspectrum

Conjecture 8.1 (Realizability image; V-1)**.** *The image of the comparison map*

$$\rho\colon\ \mathrm{Phases}_{d,G} \to \mathcal{C}_{d,G}$$

restricted to uniformly gapped lattice families is exactly the short-range-entangled subspectrum: $\operatorname{im}\rho = \mathrm{SRE}_{d,G}$ *as a subgroup of* $\pi_0\mathbf{IP}^{\mathrm{cond}}_{d,G}$*. Equivalently, an invertible abstract class is gapped-realizable if and only if it is short-range-entangled, and the non-realized classes form an identifiable obstruction subgroup*

$$\mathrm{Obs}_{d,G} \;=\; \pi_0\mathbf{IP}^{\mathrm{cond}}_{d,G}\big/\operatorname{im}\rho,$$

which contains, in the commuting-projector image, the chiral classes of Theorem 6.1.

Dim. / class	Invariant (torsion / free)	Construction	Obstruction	Realizability status
$d = 1$, on-site finite G	$H^2(G, U(1))$ (torsion); no free part	CGLW / cluster (Theorem 3.1)	none (no chiral in $1+1$d)	**Settled**: CP, complete (Theorem 5.1)
$d = 1$, fermionic $\mathbb{Z}_2$	$\mathbb{Z}_2$ (torsion)	Kitaev chain (Theorem 4.3)	none	**Settled**: CP at fixed point
$d = 2$, on-site finite G	$H^3(G, \mathbb{T})$ (torsion); no bosonic free SPT	CGLW (Theorem 3.1)	none for torsion	CP**-realized** (Theorem 3.1); completeness open (Theorem 5.2)
$d = 2$, fermionic SPT ($G_f{=}G{\times}\mathbb{Z}_2^f$)	supercohomology + beyond (torsion; e.g. $\mathbb{Z}_8$ for $G{=}\mathbb{Z}_2$)	Gu–Wen [14]; Tarantino–Fidkowski [39]; Wang–Gu [44]	none (torsion)	**Constructive**: CP for all 2+1d fermionic SPTs
$d = 2$, no symmetry (bosonic)	free part $\mathbb{Z}$, generator E_8 ($c_- = 8$)	E_8 / K-matrix (gapped, non-CP)	$c_- \neq 0$: Kapustin–Spodyneiko (Theorem 6.1(ii))	**Gapped yes,** CP **no**
$d = 2$, fermionic (class A)	free part $\mathbb{Z}$, Chern number	Chern insulator (gapped, non-CP)	$\sigma_H \neq 0$, $U(1)$: Kapustin–Fidkowski (Theorem 6.1(i))	**Gapped yes,** CP **no**
$d = 3$, on-site finite G	$H^4(G, U(1))$ (torsion) + beyond-cohomology	CGLW (in-coh.); Walker–Wang (beyond)	partial; higher chiral data	**Constructive** (in-coh. + WW); open in general
$d = 3$, beyond cohomology	cobordism classes (Kapustin)	Walker–Wang (Theorem 4.1)	chiral boundary non-CP (Theorem 4.2)	**Constructive** (bulk CP); open in general
general d, invertible SRE	$\pi_0\mathbf{IP}^{\mathrm{cond}}_{d,G}$ (torsion $\oplus$ free)	per-class (above)	σ_H/c_- no-go on free/chiral part	**Conjectural** (Section 8)

Table 1: Realizability status by spatial dimension and symmetry class. Here "CP" abbreviates commuting-projector realizability; "Gapped yes, CP no" marks classes realizable by some gapped local Hamiltonian but obstructed for commuting projectors by Theorem 6.1 — Kapustin–Fidkowski for the electric $U(1)$ response, Kapustin–Spodyneiko for the chiral central charge.

Remark 8.2. Conjecture V-1 is the essential-surjectivity statement of Theorem 2.2 made precise: it asserts that the only obstruction to realizing an invertible abstract class is that the class actually be invertible in the physical (short-range-entangled) sense, not merely formally invertible in the spectrum. It is supported by every case in the table (each realized class is SRE and each SRE class with a known construction is realized), but a proof would require the yet-unproven identification of $\mathcal{C}_{d,G}$ with $\pi_0\mathbf{IP}^{\mathrm{cond}}_{d,G}$ from Part IV (Conjecture IV-2) together with a general construction theorem, neither of which is available.

8.2 Chiral classes: gapped yes, commuting-projector no

Conjecture 8.3 (Chiral realizability; V-2)**.** *Every chiral invertible class — one with nonzero chiral central charge or nonzero Hall conductance — is gapped-realizable by a non-commuting uniformly gapped local Hamiltonian, but is never* CP*-realizable. In symbols, writing $\mathcal{C}^{\chi}_{d,G}$ for the chiral part,*

$$\mathcal{C}^{\chi}_{d,G} \subseteq \operatorname{im}\rho, \qquad \mathcal{C}^{\chi}_{d,G} \cap \operatorname{im}\rho^{\mathrm{CP}} = \{0\}.$$

Thus the no-go of Theorem 6.1 is specific to the commuting-projector class and not to gapped lattice systems: $\operatorname{im}\rho^{\mathrm{CP}}$ *is exactly the non-chiral (zero-Hall) part of* $\operatorname{im}\rho$.

Remark 8.4. The second half of Conjecture V-2 (never CP) is the theorem Theorem 6.1 for the electric/thermal Hall response; the conjectural part is that this is the *only* obstruction to CP-realizability, i.e. that every non-chiral SRE class *is* CP-realizable, and that every chiral class *is* gapped-realizable by some (necessarily non-commuting) model. The forward direction is known case by case (Chern insulators, E_8, coupled-wire constructions) but not as a theorem for all chiral classes at once. This conjecture is the precise sense in which "exactly solvable" and "physically realizable" part ways.

8.3 Realizability over profinite disorder hulls

The third conjecture is where the condensed formalism does work that ordinary topology does not: it concerns realizability not over a point but over a profinite disorder hull, as a descent statement.

Conjecture 8.5 (Profinite-family realizability; V-3)**.** *Let $\Omega = Q^{\mathbb{Z}^d}$ be a profinite disorder hull (Q a finite local-configuration set), so a disordered family of models is an Ω-point $H \in \mathfrak{Ham}_{d,G}(\Omega)$. Suppose a class $c \in \mathcal{C}_{d,G}$ is realized over a point (a translation-invariant model). Then c is realized by a* uniformly gapped Ω*-family — realizability is disorder-robust — if and only if a descent obstruction*

$$o(c) \in H^1_{\mathrm{cond}}\big(\Omega, \underline{\mathrm{Aut}}(c)\big)$$

in the relevant condensed cohomology of Ω with coefficients in the (condensed) automorphisms of the realizing model vanishes. Finite quotients of Ω detect $o(c)$: the family is uniformly-gapped-realizable iff it is realizable at every finite resolution with a common gap bound Δ.

Remark 8.6. Conjecture V-3 is the realizability analogue of the descent threads that run through the whole program (Part I's Lieb–Robinson descent, Part II's crossed-product functoriality, Part III's uniform-gap descent). Its content is that disorder-robust realizability is not a new hard analytic problem but a cohomological one *once* the point-realizability and the uniform gap (Part III) are in hand: the profinite probe Ω matches the physical configuration space exactly, and the sheaf condition over its finite quotients is what "disorder-robust" should mean. The vanishing of $o(c)$ is the obstruction to gluing finite-resolution realizations into a genuine Ω-family; when it vanishes, the crossed product $C(\Omega) \rtimes \mathbb{Z}^d$ of Part II [5, 22] carries the disorder-averaged invariant of c.

8.4 The comparison square

The three conjectures fit into one diagram, which is the realizability face of the program. Writing ρ^{CP} for the restriction to commuting-projector models and γ for group completion, the square

$$\begin{tikzcd} \{\text{CP lattice models}\}/\mathcal{W} \arrow[r, "\rho^{\mathrm{CP}}"] \arrow[d, hook] & \operatorname{im}\rho^{\mathrm{CP}} \arrow[d, hook] \\ \{\text{gapped lattice models}\}/\mathcal{W} \arrow[r, "\rho"'] & \mathcal{C}_{d,G} \end{tikzcd}$$

commutes, the left vertical is the honest inclusion of exactly-solvable models into all gapped models, and the right vertical is the strict inclusion of Theorem 6.4. Conjecture V-1 identifies the bottom image with $\mathrm{SRE}_{d,G}$; Conjecture V-2 identifies the top image with the non-chiral part; Conjecture V-3 promotes the whole square from a point to a profinite base Ω.

9 Disorder-robust realizability over profinite hulls

We expand the profinite thread, since it is the part of the realizability story that the condensed formalism is built for. Ordinary topology treats a disordered family as a section of a bundle over a compact metric space; condensed mathematics treats it as an Ω-point, where $\Omega = Q^{\mathbb{Z}^d}$ is genuinely profinite and its finite quotients are finite-resolution disorder data. The realizability question then acquires a resolution-by-resolution structure.

Definition 9.1 (Finite-resolution realizability)**.** Let $\Omega = \varprojlim_i \Omega_i$ be the inverse limit of its finite quotients. A class c is *realizable at resolution i* if there is a uniformly gapped family $H^{(i)} \in \mathfrak{Gap}_{d,G}(\Omega_i)$ whose fibrewise phase is c. It is *profinitely realizable* if there is a compatible system $(H^{(i)})_i$ with a common gap bound $\Delta > 0$, equivalently an Ω-point $H \in \mathfrak{Gap}_{d,G}(\Omega)$ with fibrewise phase c.

The content of Conjecture V-3 is that the passage from resolutionwise realizability to profinite realizability is governed by a single condensed-cohomological obstruction. Two structural facts, both consequences of the companion papers, make this plausible and are theorems modulo their stated hypotheses.

Proposition 9.2 (Uniform-gap descent enables gluing, assuming Conjecture III-2)**.** *Assume Conjecture III-2 of Part III (uniform-gap descent on the stated stability stratum). Suppose each $H^{(i)}$ is uniformly gapped with the* same *bound Δ, and that the stability hypotheses of Part III (LTQO / frustration-free with uniform constants) hold across the tower. Then the* *compatible system $(H^{(i)})_i$ defines an Ω-point of $\mathfrak{Gap}_{d,G}$, and the fibrewise phase is locally constant over Ω.*

Proof. By Part III (the uniform-gap descent, its Conjecture III-2 on the stated stratum) a family over Ω is uniformly gapped iff each finite-resolution approximation is gapped with a common Δ; the hypothesis is exactly this. The sheaf/limit description of Part I (profinite family = compatible finite data) then assembles the $H^{(i)}$ into an Ω-point of $\mathfrak{Ham}_{d,G}$, which lands in $\mathfrak{Gap}_{d,G}$ by the uniform bound. Local constancy of the phase is the Lieb–Robinson-continuity of the invariant (Part II) over the connected components of Ω's finite quotients. $\square$

Remark 9.3 (Why an obstruction can remain). Theorem 9.2 assumes a *common* gap bound and a *compatible* system. Neither is automatic: the finite-resolution realizations may exist with gaps $\Delta_i \to 0$ (not uniformly gapped, excluded from $\mathfrak{Gap}$), or may fail to be compatible across the tower (the phase at resolution i may not lift to resolution $i+1$ without a discontinuity). The first failure is an analytic one addressed by Part III's hypotheses; the second is the cohomological obstruction $o(c)$ of Conjecture V-3, living in H^1_{cond} of the hull with automorphism coefficients. This is the precise place where "realizable at every finite resolution" can fail to give "realizable over the hull," and it is a genuinely condensed phenomenon: it has no counterpart for a single translation-invariant model.

Example 9.4 (Disorder cannot create chiral realizability)**.** Disorder does not evade Theorem 6.1. If c is chiral ($\sigma_H(c) \neq 0$), then no Ω-family of commuting-projector models realizes c at any resolution, because each fibre would be a CP model with $\sigma_H = 0$; averaging over Ω cannot produce a nonzero Hall conductance from fibrewise-zero data. Thus $\operatorname{im} \rho^{\mathrm{CP}}$ over Ω is still non-chiral, consistent with Conjecture V-2. Disorder can, however, change *which* non-chiral class is realized (mobility gaps, disorder-induced transitions), and that is the content the condensed crossed-product invariant of Part II [36] is designed to track.

10 Computational verification

The accompanying Haskell package `src/bordism-realizability/` makes the two elementary propositions executable and self-checking. It is not a proof assistant; it is a battery of exact finite computations that would fail loudly if the propositions were misstated, together with QuickCheck properties that sample the parameter space.

10.1 Kitaev-chain invariant sweep

The module `Kitaev.hs` implements the Bogoliubov–de Gennes d-vector $(d_y(k), d_z(k))$ of Theorem 4.3 and computes two invariants directly from it: the winding number w, by summing signed angle increments of (d_y, d_z) around $k \in [0, 2\pi)$ on a fine grid and rounding; and the Majorana number $\mathcal{M} = \operatorname{sgn}(d_z(0)) \operatorname{sgn}(d_z(\pi))$. `Main.hs` prints an invariant table as μ sweeps across the transition at fixed $t = \Delta_{\mathrm{p}} = 1$:

> for $|\mu| < 2$ the winding is 1 and $\mathcal{M} = -1$ (topological); for $|\mu| > 2$ the winding is 0 and $\mathcal{M} = +1$ (trivial); the two invariants agree (mod 2) on the gapped locus and jump together at $|\mu| = 2$.

A second routine sweeps the full (μ, t) plane and reports the phase boundary as the locus where the winding changes, recovering $|\mu| = 2|t|$. The transition is *detected*, not hard-coded: the code locates the grid cell where the invariant changes and reports the midpoint.

10.2 Cluster-state string order

The module `Stabilizer.hs` implements an exact stabilizer-formalism engine over the binary symplectic (check-matrix) representation of Pauli operators with i-power phase tracking. It builds the cluster-state stabilizer group $\langle K_j \rangle$ of Theorem 3.3, computes $\langle P \rangle \in \{+1, -1, 0\}$ for any Pauli string P by testing membership in the group via Gaussian elimination over $\mathbb{F}_2$, and evaluates the string order parameter $S_{a,b}$ of Theorem 3.4. It confirms $\langle S_{a,b} \rangle = +1$, $\langle Z_j \rangle = \langle X_j \rangle = 0$, and computes the symmetry-breaking degradation $\langle U(\theta)^* S_{a,b} U(\theta) \rangle$ by expanding the rotated string into its 2^m Pauli terms and summing exact stabilizer expectations — recovering $(\cos\theta)^m$ without assuming the closed form.

10.3 QuickCheck properties

`Properties.hs` states one property per claim of the two propositions:

- *Invariant quantization*: for random $(\mu, t, \Delta_\mathrm{p})$ off the discriminant, the computed winding is exactly 0 or 1 and equals $\frac{1}{2}(1 - \mathcal{M})$.
- *Phase-boundary detection*: for random $t, \Delta_\mathrm{p} \neq 0$, the winding is 1 iff $|\mu| < 2|t|$, and the detected boundary matches $|\mu| = 2|t|$ within grid resolution.
- *String-order quantization*: for random even-length windows, $\langle S_{a,b} \rangle = +1$ in the cluster state and $\langle Z_j \rangle = 0$.
- *String-order degradation*: for random $\theta \in [0, \pi/2]$ and window length, the computed degraded string order equals $(\cos\theta)^m$ within tolerance and is monotone decreasing in θ.

`Main.hs` runs the demonstrations and all properties and exits nonzero on any failure, so the package doubles as a regression test of Theorems 3.4 and 4.3.

11 Discussion

11.1 What the condensed viewpoint adds here

Realizability is, at bottom, a question about explicit Hamiltonians, and one might ask what the condensed formalism contributes beyond bookkeeping. Three things. First, it gives the right home for the *disorder* thread: Conjecture V-3 and Section 9 are statements about Ω-points and descent that are awkward to phrase with ordinary bundles and natural with profinite probes. Second, it makes the comparison map ρ a morphism in a category with exact derived operations, so that "obstruction subgroup" $\mathrm{Obs}_{d,G} = \mathcal{C}_{d,G}/\operatorname{im}\rho$ is a well-behaved condensed object rather than a set-theoretic quotient. Third, it aligns realizability with the rest of the tower: the same $\mathcal{W}$, the same stabilization, the same spectrum appear in Parts III–V, so realizability is literally the surjectivity of a map the program already constructs, not a separate theory.

11.2 Limitations

We are candid about the boundaries of the rigorous content. The three Theorems are repackagings of cited results into the realizability language; their mathematical substance is due to Chen–Gu–Liu–Wen, Else–Nayak, Ogata, Kapustin–Fidkowski, and Kapustin–Spodyneiko, and we claim only the translation and the clean CP-versus-gapped framing. The two Propositions are elementary and self-contained but concern the simplest (1d, torsion) corner. Everything about the global image (Conjectures V-1, V-2, V-3) is open and depends on Part IV's still-conjectural identification of $\mathcal{C}_{d,G}$ with $\pi_0\mathbf{IP}^{\mathrm{cond}}_{d,G}$. In particular we do *not* prove that every SRE class is realizable, nor that every non-chiral class is CP-realizable; we prove the obstruction (chiral $\Rightarrow$ not CP) and organize the positive evidence.

11.3 Non-invertible order

The entire spectrum-level story is about invertible (SRE) phases. Intrinsic topological order (anyons, modular tensor categories, nonzero total quantum dimension) is not classified by $\mathbf{IP}^{\mathrm{cond}}_{d,G}$ and its realizability is a different question, requiring the condensed *higher* stack of phases and defects that Part VI flags as out of current scope. Commuting-projector models realize a great deal of non-chiral topological order (toric code, string-net / Levin–Wen, Walker–Wang), but the chiral topological orders (e.g. the non-abelian Ising phase with $c_- = 1/2$) inherit the obstruction of Theorem 6.1 through their edge and are not CP-realizable.

11.4 Open directions

The most consequential open problem is a *general construction theorem*: a procedure that, given an arbitrary SRE class in $\pi_0\mathbf{IP}^{\mathrm{cond}}_{d,G}$, outputs a uniformly gapped lattice model realizing it (non-commuting where the class is chiral). Conjecture V-1 asserts such a procedure exists; the coupled-wire and network constructions are the closest existing approximations for chiral phases. A second direction is to prove or refute the descent obstruction picture of Conjecture V-3 in a tractable family (e.g. the disordered Kitaev chain over a Bernoulli hull), where the crossed-product invariant of Part II is computable and the obstruction $o(c)$ should be explicitly checkable.

12 Conclusion

We have separated the realizability question, which abstract bordism/homotopy classes are produced by explicit gapped lattice models, into a constructive region, an obstructed region, and an open region, all inside the condensed program of this series. The constructive region is governed by two theorems: every group-cohomology class is realized by a commuting-projector model (Theorem V-A), and in $d = 1$ with on-site finite symmetry the operator-algebraic index is complete, so realized equals invariant (Theorem V-B); in $d = 2$ the index exists and every in-cohomology class is realized, though completeness is open. The obstructed region is governed by one theorem in two parts: local commuting-projector Hamiltonians have zero electric Hall conductance under $U(1)$ (Kapustin–Fidkowski) and zero chiral central charge (Kapustin–Spodyneiko), so every chiral class lies outside the commuting-projector image (Theorem V-C), and the inclusion $\operatorname{im}\rho^{\mathrm{CP}} \subsetneq \operatorname{im}\rho$ is strict wherever chiral phases exist. The open region is the global shape of the realizability image, stated as three numbered conjectures: that the image is exactly the short-range-entangled subspectrum (V-1), that chiral classes are gapped-realizable but never commuting-projector realizable (V-2), and that realizability descends over profinite disorder hulls under a condensed-cohomological obstruction (V-3). The two elementary propositions (the Kitaev chain winding invariant and the cluster-state string order) anchor the torsion corner concretely and are verified by the accompanying Haskell. Throughout, the discipline that makes the picture coherent is the distinction between realizability by *some* gapped local Hamiltonian and realizability by a *commuting-projector* model; the Kapustin–Fidkowski (electric) and Kapustin–Spodyneiko (thermal) walls are exactly the gap between them.

Code availability. The accompanying Haskell package (the Kitaev-chain invariant sweep and the exact cluster-state stabilizer engine that verify Theorems 3.4 and 4.3) is available at `github.com/YonedaAI/topological-phases-of-matter` in the directory `src/bordism-realizability/`.

References

[1] David Aasen, Zhenghan Wang, and Matthew B. Hastings. Adiabatic paths of hamiltonians, symmetries of topological order, and automorphism codes. *Physical Review B*, 106:085122, 2022.

[2] Ko Aoki. (semi)topological K-theory via solidification, 2024.

[3] Sven Bachmann, Spyridon Michalakis, Bruno Nachtergaele, and Robert Sims. Automorphic equivalence within gapped phases of quantum lattice systems. *Communications in Mathematical Physics*, 309:835–871, 2012.

[4] Agnès Beaudry, Michael Hermele, Juan Moreno, Markus J. Pflaum, Marvin Qi, and Daniel Spiegel. Homotopical foundations of parametrized quantum spin systems. *Reviews in Mathematical Physics*, 36(9):2460003, 2024.

[5] Jean Bellissard, Andreas van Elst, and Hermann Schulz-Baldes. The noncommutative geometry of the quantum hall effect. *Journal of Mathematical Physics*, 35(10):5373–5451, 1994.

[6] Chris Bourne, Johannes Kellendonk, and Adam Rennie. The K-theoretic bulk-edge correspondence for topological insulators. *Annales Henri Poincaré*, 18:1833–1866, 2017.

[7] Xie Chen, Zheng-Cheng Gu, Zheng-Xin Liu, and Xiao-Gang Wen. Symmetry protected topological orders and the group cohomology of their symmetry group. *Physical Review B*, 87:155114, 2013.

[8] Giuseppe De Nittis. Topological phases of non-interacting systems: a general approach based on states. *Letters in Mathematical Physics*, 115, 2025.

[9] Giuseppe De Nittis and Santiago G. Rendel. A scheme for topological phases of the Weyl C^*-algebra, 2026.

[10] Dominic V. Else and Chetan Nayak. Classifying symmetry-protected topological phases through the anomalous action of the symmetry on the edge. *Physical Review B*, 90:235137, 2014.

[11] Daniel S. Freed. Short-range entanglement and invertible field theories, 2014.

[12] Daniel S. Freed and Michael J. Hopkins. Reflection positivity and invertible topological phases. *Geometry & Topology*, 25:1165–1330, 2021.

[13] Davide Gaiotto and Theo Johnson-Freyd. Symmetry protected topological phases and generalized cohomology. *Journal of High Energy Physics*, 2019(5):007, 2019.

[14] Zheng-Cheng Gu and Xiao-Gang Wen. Symmetry-protected topological orders for interacting fermions: Fermionic topological nonlinear σ models and a special group supercohomology theory. *Physical Review B*, 90:115141, 2014.

[15] Po-Shen Hsin and Ryohei Kobayashi. Generalized Hall conductivities in local commuting projector models: Generalized symmetries and protected surface modes, 2025.

[16] Po-Shen Hsin and Zhenghan Wang. On topology of the moduli space of gapped hamiltonians for topological phases. *Journal of Mathematical Physics*, 64(4):041901, 2023.

[17] Anton Kapustin. Symmetry protected topological phases, anomalies, and cobordisms: Beyond group cohomology, 2014.

[18] Anton Kapustin and Lukasz Fidkowski. Local commuting projector hamiltonians and the quantum hall effect. *Communications in Mathematical Physics*, 373(2):763–769, 2019.

[19] Anton Kapustin and Nikita Sopenko. Hall conductance and the statistics of flux insertions in gapped interacting lattice systems. *Journal of Mathematical Physics*, 61(10):101901, 2020.

[20] Anton Kapustin and Nikita Sopenko. Local noether theorem for quantum lattice systems and topological invariants of gapped states. *Journal of Mathematical Physics*, 63(9):091903, 2022.

[21] Anton Kapustin and Lev Spodyneiko. Thermal Hall conductance and a relative topological invariant of gapped two-dimensional systems. *Physical Review B*, 101:045137, 2020.

[22] Johannes Kellendonk. Non commutative geometry of tilings and gap labelling. *Reviews in Mathematical Physics*, 7(7):1133–1180, 1995.

[23] Alexei Kitaev. Anyons in an exactly solved model and beyond. *Annals of Physics*, 321(1):2–111, 2006.

[24] Alexei Kitaev. Periodic table for topological insulators and superconductors. *AIP Conference Proceedings*, 1134:22–30, 2009.

[25] Alexei Yu. Kitaev. Unpaired majorana fermions in quantum wires. *Physics-Uspekhi*, 44(10S):131–136, 2001.

[26] Yosuke Kubota. Controlled topological phases and bulk-edge correspondence. *Communications in Mathematical Physics*, 349:493–525, 2017.

[27] Yosuke Kubota. Stable homotopy theory of invertible gapped quantum spin systems I: Kitaev's Ω-spectrum, 2025.

[28] Matthew Long and The YonedaAI Collaboration. Condensed locality: Lieb–robinson estimates and quasi-local dynamics on the moduli stack of hamiltonians, 2026. Part I of this series.

[29] Matthew Long and The YonedaAI Collaboration. From lattice models to effective field theories: Stabilization and the invertible condensed phase spectrum, 2026. Part IV of this series.

[30] Matthew Long and The YonedaAI Collaboration. Positivity, C*-norms, and condensed state spaces of quasi-local algebras, 2026. Part II of this series.

[31] Matthew Long and The YonedaAI Collaboration. Topological phases of matter in the condensed-mathematics paradigm: A modular research program, 2026. Part VI of this series.

[32] Matthew Long and The YonedaAI Collaboration. The uniformly gapped substack: Existence and stability of the thermodynamic spectral gap, 2026. Part III of this series.

[33] Yoshiko Ogata. Classification of gapped ground state phases in quantum spin systems, 2021. Proceedings of the International Congress of Mathematicians (ICM 2022).

[34] Yoshiko Ogata. Classification of symmetry protected topological phases in quantum spin chains, 2021.

[35] Yoshiko Ogata. A $H^3(G, \mathbb{T})$-valued index of symmetry protected topological phases with on-site finite group symmetry for two-dimensional quantum spin systems. *Forum of Mathematics, Pi*, 9:e13, 2021.

[36] Emil Prodan and Hermann Schulz-Baldes. *Bulk and Boundary Invariants for Complex Topological Insulators: From K-Theory to Physics*. Mathematical Physics Studies. Springer, 2016.

[37] Peter Scholze. Lectures on condensed mathematics, 2026. Joint work with Dustin Clausen; stable citable version of the 2019 Bonn course.

[38] W. P. Su, J. R. Schrieffer, and A. J. Heeger. Solitons in polyacetylene. *Physical Review Letters*, 42(25):1698–1701, 1979.

[39] Nicolas Tarantino and Lukasz Fidkowski. Discrete spin structures and commuting projector models for two-dimensional fermionic symmetry-protected topological phases. *Physical Review B*, 94:115115, 2016.

[40] Guo Chuan Thiang. On the K-theoretic classification of topological phases of matter. *Annales Henri Poincaré*, 17:757–794, 2016.

[41] D. J. Thouless. Quantization of particle transport. *Physical Review B*, 27(10):6083–6087, 1983.

[42] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs. Quantized hall conductance in a two-dimensional periodic potential. *Physical Review Letters*, 49(6):405–408, 1982.

[43] Kevin Walker and Zhenghan Wang. (3+1)-TQFTs and topological insulators. *Frontiers of Physics*, 7:150–159, 2012.

[44] Qing-Rui Wang and Zheng-Cheng Gu. Construction and classification of symmetry-protected topological phases in interacting fermion systems. *Physical Review X*, 10:031055, 2020.

[45] Charles Zhaoxi Xiong. Minimalist approach to the classification of symmetry protected topological phases. *Journal of Physics A: Mathematical and Theoretical*, 51(44):445001, 2018.