From Lattice Models to Effective Field Theories: Stabilization and the Invertible Condensed Phase Spectrum

Matthew Long
The YonedaAI Collaboration
YonedaAI Research Collective
Chicago, IL
matthew@yonedaai.com · https://yonedaai.com

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Abstract

We study the fourth analytic input to the condensed-mathematics program for topological phases of matter: the relationship between microscopic lattice systems and their effective field theories. Over the stabilized phase ∞-groupoid $\mathfrak{Phase}_{d,G}$ built in Part III we make the stacking operation $\boxtimes$ explicit as a symmetric-monoidal structure and record what follows formally. On components, stacking makes the set of phases a commutative monoid whose unit is the trivial product state and whose invertible elements are exactly the short-range-entangled (SRE) phases. Group completion of this monoid is governed by a universal property that we prove, and its explicit presentation isolates a *stabilization element* whose physical content is the addition of trivial ancillas. Restricting to the invertible sector and invoking the recognition principle for group-like E_∞-spaces yields a connective spectrum, the invertible condensed phase spectrum $\mathbf{IP}^{\mathrm{cond}}_{d,G}$; Kubota's operator-algebraic Ω-spectrum is a rigorous carrier for its homotopy groups, and Aoki's solidification bridge supplies the analytic completion that a condensed refinement requires. Against this we set the effective-field-theory (bordism / Anderson-dual) classification of Freed–Hopkins. The organizing claim of the program, that the microscopic and field-theoretic classifications agree, is stated as a comparison conjecture at three increasingly strong levels: on π_0 of phase sets, on spectra, and on families over condensed probes. We do not prove it. We assemble the evidence that constrains it: in $d = 1$ Ogata's operator-algebraic index is a complete invariant and the π_0 comparison is a theorem, and for free fermions the tenfold-way K-theory of Kitaev realizes it, with the 8-fold real and 2-fold complex Bott periodicities verified by machine. Renormalization is framed as a filtered / pro-completion, and we conjecture that the lattice-to-EFT passage is computed by solidification, that solidification is compatible with group completion, and that the relative transition charge is the boundary map of the cofiber sequence for $\mathbf{IP}^{\mathrm{cond}}$. Two honest limitations bound the discussion: the equivalence is expected only under short-range-entanglement hypotheses, and noninvertible topological order lies outside the invertible spectrum entirely. Accompanying Haskell code implements finitely presented commutative monoids, their group completion, and the tenfold-way table, with QuickCheck tests of the monoid laws, the universal property, and Bott periodicity.

Contents

1 Introduction

This is the fourth paper of a six-part series that reorganizes the theory of topological phases of matter inside condensed mathematics. The program builds a single tower,

$$\text{local interactions} \rightsquigarrow \mathfrak{Ham}_{d,G} \rightsquigarrow \mathfrak{Gap}_{d,G} \rightsquigarrow \mathfrak{Phase}_{d,G} \rightsquigarrow \mathbf{IP}^{\mathrm{cond}}_{d,G}, \tag{1}$$

whose floors are, respectively: the condensed moduli stack of G-symmetric quasi-local Hamiltonians (Part I, [21]); its uniformly gapped substack, with the observable side and its solid K-theory installed in Part II, [23]; the stabilized phase ∞-groupoid whose existence and robustness rest on the gap-stability results of Part III, [25]; and, at the top, a connective spectrum assembled from the invertible sector. The present paper is about the top two floors of (1) and, in particular, about the bridge from a lattice system to its effective field theory (EFT).

1.1 The lattice–EFT problem

A gapped lattice Hamiltonian is a microscopic object: finite-dimensional Hilbert spaces on sites, a local interaction, an energy gap above the ground state. Its low-energy physics is described, when the system is short-range entangled, by an *invertible* topological quantum field theory: an EFT with a one-dimensional state space on every closed spatial slice [10, 11]. Two classification schemes then coexist:

- the *microscopic* one, which sorts lattice systems into phases by adiabatic / finite-depth equivalence and stabilization, producing the set $\mathrm{Phases}_{d,G} = \pi_0\,\mathrm{Shape}(\mathfrak{Phase}_{d,G})$; and
- the *field-theoretic* one, which sorts invertible TQFTs into deformation classes, producing (under reflection positivity) a bordism / Anderson-dual invariant computed by a spectrum $\mathbf{I}_{\mathrm{FH}}$ [11, 15, 12, 35].

The fourth item on the program's list of analytic obligations ([21], §1.8) is to control the relationship between these two. We take "equivalence between microscopic lattice systems and effective field theories" to mean the assertion that the two classifications agree, and we take seriously the fact that this is, at present, *open*. It is the organizing conjecture of the paper, not a theorem we are going to prove.

1.2 What is proved, and what is conjectured

The paper keeps a strict line between the two. Provable today, from cited present-day results, are the algebraic and homotopical facts that make the top of the tower well-posed:

- stacking $\boxtimes$ endows $\mathrm{Phases}_{d,G}$ with the structure of a commutative monoid (Proposition 3.1); the trivial phase is the unit and the SRE phases are the invertible elements;

- the Grothendieck group completion of a commutative monoid exists and is characterized by a universal property (Theorem IV-A, Theorem 4.2), whose explicit form exposes the "stabilization element" behind physical stabilization by ancillas (Proposition 4.3);
- the invertible sector is grouplike, hence—by the recognition principle for grouplike E_∞-spaces used in this setting by [4, 20]—the infinite-loop space of a connective spectrum, for whose homotopy groups Kubota constructs a rigorous Ω-spectrum carrier (Theorem IV-B, Theorem 5.3);
- the free-fermion tenfold way is 8-fold (2-fold) Bott periodic in the real (complex) classes (Proposition 8.3); and
- the 2D Hall conductance of an SRE state is an integer-valued phase invariant, locally computable, with a higher Berry generalization for families (Theorem IV-C, Theorem 9.1).

Everything past this is stated as a numbered *Conjecture* with a stable identifier of the form IV-n. Five appear: the condensed / solid refinement of the spectrum (IV-1), the lattice–EFT comparison equivalence (IV-2, the organizing conjecture), the identification of the relative transition charge with a spectral boundary map (IV-3), the compatibility of solidification with group completion (IV-4), and the existence of a renormalization functor computing the EFT limit (IV-5).

1.3 Three levels of equivalence

The word "equivalence" hides a hierarchy that we insist on making visible. There are (at least) three honestly different statements one might mean:

(L0) **Sets of phases.** A bijection $\mathrm{Phases}_{d,G} \cong \pi_0 \mathbf{I}_{\mathrm{FH}}$ of abelian groups: the microscopic and field-theoretic invariants classify the same phases.

(L1) **Spectra.** An equivalence $\mathbf{IP}^{\mathrm{cond}}_{d,G} \simeq \mathbf{I}_{\mathrm{FH}}$ (after a specified completion) of connective spectra: not only π_0 but the entire homotopy type—higher pumps and families—match.

(L2) **Families.** A natural equivalence over condensed probes S, so that the comparison respects continuous families, profinite disorder hulls $\Omega = Q^{\mathbb{Z}^d}$, and symmetry data simultaneously.

These are strictly increasing in strength: (L2)⇒(L1)⇒(L0). Existing anchors settle (L0) in favourable cases (Ogata in $d = 1$; free fermions in every d) and give rigorous *targets* for (L1) (Kubota's Ω-spectrum). Nothing in the literature establishes (L2), and the condensed refinement it would require is exactly Conjecture IV-1. Keeping the levels apart is not pedantry: a proof at (L0) leaves (L1) and (L2) untouched, and most of what one wants the tower (1) for (parametrized invariants, descent along disorder hulls, the transition calculus) lives at (L1) and (L2).

1.4 Relation to companion papers

This paper consumes the outputs of Parts I–III and feeds Parts V–VI.

Part I (Condensed Locality [21]*)* makes the interaction space a Banach space $\mathcal{B}_F$ and the Heisenberg dynamics a morphism of condensed objects; it is the reason a "family of systems" over a probe S is a well-defined thing to stack. *Part II (Positivity and C^*-Norms* [23]*)* attaches the condensed observable algebra $\underline{A}$ and, through Aoki's theorem $K_{\mathrm{op}}(A) \simeq \mathrm{Solid}(K_{\mathrm{alg}}(\underline{A}))$, the solid K-theory that we use here as the analytic completion step in the EFT passage (Section 6). *Part III (The Uniformly Gapped Substack* [25]*)* is what makes π_0 of the stabilized stack a robust invariant in the first place; without the stability of the gap under quasi-local perturbations, the monoid $\mathrm{Phases}_{d,G}$ would not be well-defined, and the undecidability wall of Cubitt–Pérez-García–Wolf [7] forbids us from pretending otherwise. *Part V (Physical Realizability* [22]*)* takes over exactly where our comparison conjecture ends: it asks which classes in $\mathbf{IP}^{\mathrm{cond}}_{d,G}$ are realized by honest lattice families, a question sharpened by the Kapustin–Fidkowski no-go [16]. *Part VI (Synthesis* [24]*)* assembles all five modules and states the global Master Conjecture; our Conjecture IV-2 is its top floor. We import notation verbatim from the series' canonical conventions and do not restate it.

1.5 Outline

Section 2 recalls the stabilized stack and fixes the stacking product. Sections 3 and 4 carry the provable algebra: the symmetric-monoidal structure and its group completion. Section 5 assembles the invertible condensed phase spectrum. Section 6 formulates renormalization as a pro-completion and states the solidification conjectures. Section 7 states the comparison conjecture at all three levels, and Section 8 marshals the evidence: $d = 1$ completeness and free-fermion Bott periodicity. Section 9 treats concrete lattice invariants, the SSH example, and transitions. Section 10 describes the accompanying machine verification, and Sections 11 and 12 discuss limitations and conclude.

2 The stabilized phase stack and its stacking product

We recall only what is needed and refer to [25] for the analytic construction. Fix a spatial dimension d, a lattice L, on-site Hilbert spaces, an F-function locality class, and a symmetry group G with condensation $\underline{G}$.

Definition 2.1 (the tower, [21, 25])**.** Write $\mathfrak{Ham}_{d,G}$ for the condensed moduli stack of G-symmetric quasi-local Hamiltonians; on a profinite probe S its points are S-continuous families of admissible interactions. Its *uniformly gapped* substack is

$$\mathfrak{Gap}_{d,G}(S) = \bigcup_{\Delta > 0} \Big\{ H \in \mathfrak{Ham}_{d,G}(S) : \inf_{s \in S} \mathrm{gap}(H_s) \geq \Delta \Big\}.$$

Let $\mathcal{W}$ be the class of gapped adiabatic / finite-depth quasi-local equivalences to be inverted, and $(-)^{\mathrm{st}}$ stabilization by trivial product-state ancillas. The *stabilized phase ∞-groupoid* is

$$\mathfrak{Phase}_{d,G} := \big(\mathfrak{Gap}_{d,G}[\mathcal{W}^{-1}]\big)^{\mathrm{st}}, \qquad \mathrm{Phases}_{d,G} := \pi_0 \, \mathrm{Shape}(\mathfrak{Phase}_{d,G}).$$

The word *uniformly* is essential: a family in which each H_s is gapped, but with $\inf_s \mathrm{gap}(H_s) = 0$, is excluded, and this uniformity is what Part III's stability theorems protect under perturbation. We take from [25] that $\mathrm{Phases}_{d,G}$ is well defined on the stratum where the stability hypotheses hold with uniform constants; every statement below is made there.

Remark 2.2 (slogan). We use the program's boxed slogan without alteration:

> *A topological phase is a component of the stabilized condensed stack of gapped systems.*

The π-dictionary is likewise fixed: π_0 = phases; π_1 = adiabatic pumps and phase automorphisms; π_n = higher families and higher defects.

We now fix the operation the rest of the paper is about.

Definition 2.3 (stacking)**.** Let $H^{(1)}, H^{(2)}$ be admissible systems on L with on-site spaces $\{h_x^{(1)}\}, \{h_x^{(2)}\}$ and interactions $\Phi^{(1)}, \Phi^{(2)}$. Their *stack* $H^{(1)} \boxtimes H^{(2)}$ is the system on L with on-site spaces $\{h_x^{(1)} \otimes h_x^{(2)}\}$ and interaction

$$(\Phi^{(1)} \boxtimes \Phi^{(2)})(X) = \Phi^{(1)}(X) \otimes \mathbf{1} + \mathbf{1} \otimes \Phi^{(2)}(X).$$

The *trivial* system $\mathbf{1}$ has one-dimensional on-site spaces and zero interaction. Having no excited states at all, it is assigned gap $+\infty$ by convention, so that stabilization by $\mathbf{1}$ never lowers a gap (Lemma 2.4). Stacking of G-symmetric systems is G-symmetric under the diagonal action.

Lemma 2.4 (stacking preserves the gapped substack)**.** *If $H^{(1)}, H^{(2)}$ have unique gapped ground states with gaps $\Delta_1, \Delta_2 > 0$, then $H^{(1)} \boxtimes H^{(2)}$ has a unique gapped ground state with* $\mathrm{gap}(H^{(1)} \boxtimes H^{(2)}) = \min(\Delta_1, \Delta_2)$. *Consequently $\boxtimes$ restricts to a map $\mathfrak{Gap}_{d,G} \times \mathfrak{Gap}_{d,G} \to \mathfrak{Gap}_{d,G}$, and if both factors are uniformly gapped over a probe S with bounds Δ_1, Δ_2, the stack is uniformly gapped with bound* $\min(\Delta_1, \Delta_2)$.

Proof. On a finite volume V the tensor-sum Hamiltonian $H_V^{(1)} \otimes \mathbf{1} + \mathbf{1} \otimes H_V^{(2)}$ has spectrum $\{\lambda + \mu\}$ with $\lambda \in \mathrm{spec}(H_V^{(1)})$, $\mu \in \mathrm{spec}(H_V^{(2)})$. Its ground energy is $E_0^{(1)} + E_0^{(2)}$ with the product ground state, unique because each factor's is; the first excited energy is $\min\big(E_1^{(1)} + E_0^{(2)}, E_0^{(1)} + E_1^{(2)}\big)$, so the gap is $\min(\Delta_1, \Delta_2)$. (If a factor has no excited state at all—the trivial system $\mathbf{1}$, with gap $+\infty$—the corresponding term is absent and the minimum returns the other gap, so stacking with $\mathbf{1}$ leaves the gap unchanged.) These estimates are uniform in V and, taking infima over $s \in S$, uniform over the probe. The thermodynamic statement follows from the infinite-volume GNS construction of [25]; the minimum of two positive uniform bounds is a positive uniform bound. $\square$

That $\boxtimes$ also descends through $[\mathcal{W}^{-1}]$ and $(-)^{\mathrm{st}}$ is where the symmetric-monoidal bookkeeping begins, and is the subject of the next section.

3 Stacking as a symmetric monoidal structure

The physics of stacking is old: put two systems side by side without coupling them. The mathematics we need is the statement that this operation is symmetric monoidal on the phase stack and, in particular, makes π_0 a commutative monoid. At the level of the full ∞-stack the symmetric monoidal structure is part of the program (it uses the higher stack structure of $\mathfrak{Ham}_{d,G}$, itself a conjecture of Part I); the rigorous instances for parametrized spin systems are [4]. At the level of π_0, however, the monoid laws are elementary and we verify them outright.

Proposition 3.1 (commutative monoid of phases)**.** *The operation induced by* $\boxtimes$ *makes* $(\mathrm{Phases}_{d,G}, \boxtimes, [\mathbf{1}])$ *a commutative monoid: it is associative and commutative up to the equivalences in* $\mathcal{W}$*, with two-sided unit the class* $[\mathbf{1}]$ *of the trivial product state.*

Proof. By Lemma 2.4, $\boxtimes$ sends pairs of uniformly gapped families to uniformly gapped families, so it descends to a binary operation on $\mathfrak{Gap}_{d,G}$. We must check it respects $\mathcal{W}$ and $(-)^{\mathrm{st}}$ and satisfies the laws on classes.

Respect for $\mathcal{W}$. If $\gamma_i : H^{(i)} \rightsquigarrow K^{(i)}$ are gapped adiabatic / finite-depth equivalences, then $\gamma_1 \otimes \gamma_2$ is a finite-depth quasi-local equivalence $H^{(1)} \boxtimes H^{(2)} \rightsquigarrow K^{(1)} \boxtimes K^{(2)}$: a finite-depth circuit tensor a finite-depth circuit is finite-depth, and the quasi-adiabatic continuation of a tensor-sum path is the tensor of the continuations ([13, 3]). Hence $\boxtimes$ descends to $\mathfrak{Gap}_{d,G}[\mathcal{W}^{-1}]$ and, since stacking with $\mathbf{1}$ adds trivial ancillas, commutes with $(-)^{\mathrm{st}}$.

Associativity and commutativity. On on-site spaces, $\boxtimes$ is the tensor product of Hilbert spaces, which is associative and commutative up to the canonical associator and the swap $\sigma_{h,h'} : h \otimes h' \to h' \otimes h$. The tensor-sum interaction of Definition 2.3 is symmetric in its two arguments under σ, and the associator / swap are finite-depth (indeed depth-zero, on-site) quasi-local isomorphisms, hence lie in $\mathcal{W}$. (For fermionic systems the on-site spaces are super-vector spaces and σ carries the Koszul sign; this changes none of the above, as σ is still a depth-zero on-site isomorphism, and the symmetric-monoidal structure becomes the graded one.) Therefore $[H^{(1)} \boxtimes (H^{(2)} \boxtimes H^{(3)})] = [(H^{(1)} \boxtimes H^{(2)}) \boxtimes H^{(3)}]$ and $[H^{(1)} \boxtimes H^{(2)}] = [H^{(2)} \boxtimes H^{(1)}]$ in $\mathrm{Phases}_{d,G}$.

Unit. Stacking with $\mathbf{1}$ replaces each h_x by $h_x \otimes \mathbb{R} \cong h_x$ and adds nothing to the interaction, giving a depth-zero equivalence $H \boxtimes \mathbf{1} \rightsquigarrow H$ in $\mathcal{W}$. Thus $[\mathbf{1}]$ is a two-sided unit. The three laws are precisely the axioms of a commutative monoid. □

Proposition 3.2 (invertible elements are the SRE phases)**.** *An element* $[H] \in \mathrm{Phases}_{d,G}$ *is invertible for* $\boxtimes$ *if and only if there is a system* $\bar{H}$ *with* $[H] \boxtimes [\bar{H}] = [\mathbf{1}]$*, i.e.* H *is* short-range entangled*: stacked with a partner it becomes trivial after stabilization. Write* $\mathfrak{Phase}^{\times}_{d,G} \subseteq \mathfrak{Phase}_{d,G}$ *for the full sub-*∞*-groupoid on the invertible objects and* $\mathrm{Phases}^{\times}_{d,G}$ *for its* π_0.

Proof. This is the definition of an invertible element in a monoid, transported through π_0. That the invertible objects form a full symmetric-monoidal sub-∞-groupoid is formal: invertibility is preserved by $\boxtimes$ (a tensor of invertibles is invertible with inverse the tensor of inverses) and detected on π_0; the equivalences in $\mathcal{W}$ preserve it. □

The candidate partner $\bar{H}$ is, physically, the orientation-reversed or complex-conjugate system; that every SRE phase actually admits such an inverse is the content of the invertibility of the low-energy TQFT [10, 11] and is used, not proved, here. The distinction between the full monoid $\mathrm{Phases}_{d,G}$ and its invertible submonoid $\mathrm{Phases}^{\times}_{d,G}$ is the distinction between "all gapped phases" (including intrinsic topological order, which has no inverse) and the SRE phases that an invertible EFT can see. The spectrum of Section 5 is built from the latter; the group completion of Section 4 is what the former needs.

4 Group completion and the stabilization element

A commutative monoid is not a group: some phases have no inverse, and even among those that "cancel" the cancellation may require adding a common summand. The universal way to force inverses is group completion, and its explicit form is exactly where the physics of stabilization by ancillas becomes visible. Nothing in this section is new mathematics: it is the Grothendieck construction, but the reading of the stabilization element is the point.

Definition 4.1 (group completion)**.** Let $(M, +, 0)$ be a commutative monoid. Its *group completion* (Grothendieck group) is $K(M) := (M \times M)/\!\sim$, where $(a, b) \sim (a', b')$ iff there is $e \in M$ with

$$a + b' + e \;=\; a' + b + e.$$

Addition is $[(a, b)] + [(c, d)] = [(a + c, b + d)]$, and $\gamma_M : M \to K(M)$, $\gamma_M(m) = [(m, 0)]$.

Theorem 4.2 (**IV-A**: universal property of group completion)**.** *For any commutative monoid M:*

(i) $\sim$ is a congruence and $K(M)$ is an abelian group, with identity $[(0, 0)]$ and $-[(a, b)] = [(b, a)]$;

(ii) γ_M is a monoid homomorphism;

(iii) for every abelian group A and monoid homomorphism $f : M \to A$ there is a unique group homomorphism $\bar{f} : K(M) \to A$ with $\bar{f} \circ \gamma_M = f$.

The pair $(K(M), \gamma_M)$ is thereby determined up to unique isomorphism.

Proof. *(i)* Reflexivity and symmetry are immediate. For transitivity, suppose $(a, b) \sim (c, d)$ via e and $(c, d) \sim (g, h)$ via e', so $a + d + e = c + b + e$ and $c + h + e' = g + d + e'$. Put $e'' = d + e + e'$. Then

$$\begin{aligned} a + h + e'' &= (a + d + e) + h + e' = (c + b + e) + h + e' = b + e + (c + h + e') \\ &= b + e + (g + d + e') = g + b + e'', \end{aligned}$$

so $(a, b) \sim (g, h)$. If $(a, b) \sim (a', b')$ via e then adding $c+d$ shows $(a+c, b+d) \sim (a'+c, b'+d)$ via the same e, so addition is well defined; it inherits associativity, commutativity, and the identity $[(0, 0)]$ from M. Finally $[(a, b)] + [(b, a)] = [(a + b, a + b)] = [(0, 0)]$ because $(a + b, a + b) \sim (0, 0)$ via $e = 0$. Hence $K(M)$ is an abelian group.

(ii) $\gamma_M(m) + \gamma_M(n) = [(m + n, 0)] = \gamma_M(m + n)$ and $\gamma_M(0) = [(0, 0)]$.

(iii) Set $\bar{f}([(a, b)]) := f(a) - f(b) \in A$. This is well defined: if $(a, b) \sim (a', b')$ via e then $f(a) + f(b') + f(e) = f(a') + f(b) + f(e)$ in A, and A is a group, so $f(a) - f(b) = f(a') - f(b')$. It is a homomorphism by construction and $\bar{f}(\gamma_M(m)) = f(m) - f(0) = f(m)$. For uniqueness, note $[(a, b)] = \gamma_M(a) - \gamma_M(b)$ in $K(M)$; any g with $g \circ \gamma_M = f$ therefore satisfies $g([(a, b)]) = f(a) - f(b) = \bar{f}([(a, b)])$. □

The construction is functorial: a monoid homomorphism $\varphi : M \to N$ induces $K(\varphi) : K(M) \to K(N)$, and K is left adjoint to the forgetful functor from abelian groups to commutative monoids. We will use this adjunction when discussing whether solidification commutes with group completion (Conjecture IV-4).

Now the element e. Its presence is the whole difference between a group and a monoid, and it is the algebraic shadow of ancilla stabilization.

Proposition 4.3 (the stabilization element)**.** *For $a, b \in M$, $\gamma_M(a) = \gamma_M(b)$ in $K(M)$ if and only if there is $e \in M$ with $a + e = b + e$. Consequently γ_M is injective iff M is cancellative, and in general the largest quotient of M on which γ_M is injective is $M^{\mathrm{st}} := M/{\approx}$, where $a \approx b$ iff $a + e = b + e$ for some e. The monoid M^{st} is cancellative and $K(M) \cong K(M^{\mathrm{st}})$.*

Proof. $\gamma_M(a) = \gamma_M(b)$ means $(a, 0) \sim (b, 0)$, i.e. $a + 0 + e = b + 0 + e$ for some e; that is $a + e = b + e$. Injectivity of γ_M is then the implication $a + e = b + e \Rightarrow a = b$, i.e. cancellativity. The relation $\approx$ is a congruence (if $a + e = b + e$ then $a + c + e = b + c + e$ for every c), and M^{st} is cancellative because, for an *arbitrary* $[c] \in M^{\mathrm{st}}$, $[a] + [c] = [b] + [c]$ in M^{st} means $a + c + e' = b + c + e'$ for some e', hence $a + e'' = b + e''$ with $e'' = c + e'$, i.e. $a \approx b$. Since $\approx$ is exactly the kernel pair of γ_M, the map $M^{\mathrm{st}} \to K(M)$ is injective and induces $K(M^{\mathrm{st}}) \xrightarrow{\cong} K(M)$ by the universal property. □

Remark 4.4 (why $(-)^{\mathrm{st}}$ is in Definition 2.1, and what it is not). Proposition 4.3 is the reason the stabilization $(-)^{\mathrm{st}}$ appears in the definition of $\mathfrak{Phase}_{d,G}$: two gapped systems are declared the same phase when they agree *after adding trivial product-state ancillas*. This is a genuine operation, not stacking with the unit $\mathbf{1}$. A product state on higher-dimensional on-site spaces is not $\mathcal{W}$-equivalent to $\mathbf{1}$ (finite-depth quasi-local circuits preserve on-site Hilbert-space dimension, so they cannot shrink an ancilla back to a point), and inverting "$\boxtimes$ a trivial product state" therefore enlarges the equivalence in a way stacking with $\mathbf{1}$ never could. Write $\approx_{\mathrm{prod}}$ for the resulting relation ($a \approx_{\mathrm{prod}} b$ iff $a+e = b+e$ for some *product state* e); the physically stabilized phase monoid is $M/{\approx_{\mathrm{prod}}}$.

Two cautions keep this honest. First, $\approx_{\mathrm{prod}}$ is in general *finer* than the full stable equality $\approx$ of Proposition 4.3, which allows an *arbitrary* $e \in M$: when intrinsic topological order is present, $a+m = b+m$ for a non-invertible m need not entail $a+e = b+e$ for any product state e. So $M/{\approx_{\mathrm{prod}}}$ need not be cancellative, and the group completion K can collapse it *further*; the natural map $M/{\approx_{\mathrm{prod}}} \to K(\mathrm{Phases}_{d,G})$ is then not injective. This is the "collapse" discussed in Section 11, and it is why we do *not* claim the full phase monoid embeds in its group completion. Second, on the *invertible* submonoid the two relations coincide: if a is invertible with partner $\bar{a}$, then $a \boxtimes \bar{a}$ is stably a product state, so any witness e of $a+e = b+e$ can be traded for a product state after stacking with enough copies of $\bar{a}$. A grouplike monoid is already cancellative and equals its own group completion, so $\mathrm{Phases}^{\times}_{d,G} \cong K(\mathrm{Phases}^{\times}_{d,G})$ is an abelian group with no further completion needed, which is exactly why the spectrum of Section 5 is built from the invertible sector, the one place where stabilization and group completion agree.

Example 4.5 (a non-cancellative toy)**.** Let $M = \langle a, t \mid t + a = t \rangle$ be the commutative monoid on generators a, t with the single relation $t + a = t$ (a generator t that absorbs a). Here $a \neq 0$ in M, yet $\gamma_M(a) = \gamma_M(0)$ because $a + t = 0 + t$: the stabilization element is

$e = t$. So a becomes invisible in $K(M)$—the algebra behind “a phase nontrivial on the nose but trivial once enough ancillas are added.” This does *not* make the completion vanish: no relation forces t to be absorbed—the only relation is $t + a = t$, and $nt + e = e$ has no solution in M for $n \geq 1$—so the multiples of t are cancellative among themselves, and

$$K(M) \;\cong\; \mathbb{Z}, \qquad \text{generated by } [t], \quad \text{with } [a] = 0.$$

The example is non-cancellativity in its purest form—one generator collapses under stabilization while another survives. The accompanying code (Section 10) computes $K(M)$ for this M and confirms both facts: $[a] = 0$, while the multiples $[nt]$ $(n \geq 1)$ are pairwise distinct.

5 The invertible condensed phase spectrum

We now assemble the top floor of (1). The mechanism is the standard one that turns a grouplike commutative monoid object in spaces into a connective spectrum; what is specific to our setting is the identification of the input with the invertible phase groupoid and the (conjectural) condensed refinement.

5.1 From the invertible sector to a connective spectrum

By Proposition 3.2 the invertible sector $\mathfrak{Phase}^{\times}_{d,G}$ is a symmetric-monoidal sub-∞-groupoid of $\mathfrak{Phase}_{d,G}$ in which every object is $\boxtimes$-invertible; that is, a *Picard ∞-groupoid*, equivalently a grouplike E_∞-space.

Theorem 5.1 (recognition, conditional)**.** *Suppose $\mathfrak{Phase}^{\times}_{d,G}$ is a Picard ∞-groupoid (a grouplike E_∞-space). Then there is a connective spectrum $\mathbf{IP}^{\mathrm{cond}}_{d,G}$, unique up to equivalence, with $\Omega^\infty \mathbf{IP}^{\mathrm{cond}}_{d,G} \simeq \mathfrak{Phase}^{\times}_{d,G}$, and*

$$\pi_0 \mathbf{IP}^{\mathrm{cond}}_{d,G} \cong \mathrm{Phases}^{\times}_{d,G}, \qquad \pi_n \mathbf{IP}^{\mathrm{cond}}_{d,G} \cong \pi_n\big(\mathfrak{Phase}^{\times}_{d,G}\big) \ (n \geq 1).$$

In particular π_1 records adiabatic pumps / phase automorphisms and π_n higher families, per the π-dictionary.

Proof. This is the recognition principle for grouplike E_∞-spaces: the ∞-category of connective spectra is equivalent to that of grouplike E_∞-spaces via Ω^∞, with inverse the connective-spectrum functor. The hypothesis is that $\mathfrak{Phase}^{\times}_{d,G}$ is such an object. We do not reprove the recognition principle; in the precise setting of quantum lattice systems the passage from the invertible / grouplike space of gapped systems to a spectrum is carried out rigorously in [4] (parametrized foundations) and [20] (an explicit Ω-spectrum). We record the statement conditionally because the hypothesis—that $\mathfrak{Phase}_{d,G}$ is a symmetric-monoidal ∞-groupoid with the invertible sector grouplike—rests on the higher-stack structure of $\mathfrak{Ham}_{d,G}$, which is Conjecture I-1 of [21]. □

Definition 5.2 (invertible condensed phase spectrum)**.** $\mathbf{IP}^{\mathrm{cond}}_{d,G}$ is the connective spectrum of Theorem 5.1, the *invertible condensed phase spectrum*. Its defining property is $\Omega^\infty \mathbf{IP}^{\mathrm{cond}}_{d,G} \simeq \mathfrak{Phase}^{\times}_{d,G}$.

The name contains a promissory note: as constructed, $\mathbf{IP}^{\mathrm{cond}}_{d,G}$ is a spectrum of *spaces* (the shape of the invertible condensed groupoid), and whether it genuinely lifts to a spectrum object internal to condensed / solid mathematics is Conjecture IV-1 below. We keep the notation $\mathbf{IP}^{\mathrm{cond}}$ throughout and flag the conjectural status wherever it matters.

5.2 A rigorous carrier for the homotopy groups

The value of Theorem 5.1 is limited by its hypothesis. What rescues it from vacuity is that its output has an independently constructed, fully rigorous model.

Theorem 5.3 (**IV-B**: Kubota's Ω-spectrum, [20]). *There is an Ω-spectrum IP^*, built from the operator-algebraic formulation of invertible gapped quantum spin systems, whose homotopy groups are the groups of invertible gapped spin systems in each dimension (with variants for crystallographic symmetry). It realizes Kitaev's proposal [19] that invertible phases are the homotopy groups of a spectrum.*

This is a cited theorem, not ours. Its relationship to Definition 5.2 is the subject of the next two conjectures: IP^* is the smooth / operator-algebraic avatar of the spaces underlying $\mathbf{IP}^{\mathrm{cond}}_{d,G}$, and the content of Conjecture IV-1 is that $\mathbf{IP}^{\mathrm{cond}}$ is its condensed refinement.

Conjecture (**IV-1**: condensed / solid refinement). $\mathbf{IP}^{\mathrm{cond}}_{d,G}$ *is a connective spectrum object internal to solid modules (equivalently, a condensed connective spectrum) whose underlying spectrum of spaces—its shape, obtained by forgetting the solid structure—is a connective cover of Kubota's IP^* [20], and whose homotopy solid modules refine the homotopy groups of IP^*. The refinement is the one carried by the solid K-theory of the condensed observable algebra of Part II [23], via Aoki's identification $K_{\mathrm{op}}(A) \simeq \mathrm{Solid}(K_{\mathrm{alg}}(\underline{A}))$ [2].*

The reason to want IV-1, rather than to be content with Kubota's smooth spectrum, is uniformity across the program: the profinite-disorder hulls $\Omega = Q^{\mathbb{Z}^d}$, the continuous families over condensed probes, and the analytic completions all live naturally in condensed / solid mathematics [31, 30, 6], and a spectrum internal to that world is what makes level (L2) of Section 1.3 even statable. It is, however, a genuine conjecture: no condensed enhancement of IP^* exists in the literature.

6 Renormalization and the effective-field-theory passage

Between a lattice system and its EFT sits renormalization: coarse-grain, discard short-distance data, iterate, take a limit. We formulate this as a filtered / pro-system and identify its analytic limit with solidification. Both statements are conjectural; the point is to say precisely *what* would have to be true. The two conjectures of this section carry the stable series identifiers IV-5 and IV-4: like every IV-n label in this paper they are fixed across the six-part series and referenced by the same name elsewhere, so they are not ordered by first appearance here (IV-2, the central comparison conjecture, follows in Section 7).

6.1 Coarse-graining as a filtered system

Definition 6.1 (block-spin coarse-graining)**.** A *coarse-graining* of scale ℓ is a map $R_\ell : \mathfrak{Ham}_{d,G} \to \mathfrak{Ham}_{d,G}$ that partitions L into blocks of diameter ℓ, replaces the on-site space of a block by a chosen subspace (a truncation or isometry V_ℓ), and pushes the interaction forward, $\Phi \mapsto V_\ell^* \Phi V_\ell$, retaining the induced quasi-local structure. Composition of scales gives a filtered system $\cdots \to R_{\ell_2} \to R_{\ell_1} \to \mathrm{id}$ indexed by the poset of scales.

The finite-resolution truncations of a profinite disorder hull $\Omega = \varprojlim \Omega_i$ ([23], §1.7) are the disorder-theoretic instance of exactly this poset: coarse-graining in space and coarsening the configuration data are the same kind of pro-operation. This is why the EFT limit and the descent along Ω share a formalism.

Conjecture (**IV-5**: renormalization functor)**.** *The coarse-grainings $\{R_\ell\}$ assemble into a filtered system of endofunctors of $\mathfrak{Gap}_{d,G}$ compatible with $\boxtimes$ (a lax symmetric-monoidal pro-endofunctor), whose limit / colimit*

$$R_\infty := \mathrm{colim}_\ell\, R_\ell$$

exists on the SRE stratum and computes the effective field theory: for an SRE family H, $R_\infty H$ is (the lattice model of) the fixed-point theory whose deformation class is the EFT invariant. The induced map on phases $\mathrm{Phases}_{d,G} \to \mathrm{Phases}_{d,G}$ is idempotent with image the SRE sub-monoid.

Coarse-graining is delicate for the same reason the gap is: the undecidability of the spectral gap [7] forbids any general algorithm certifying that a coarse-graining flows to a gapped fixed point, so IV-5 is asserted on the SRE stratum only, and even there its content is the existence of the limit, not a procedure to compute it. This mirrors the conditional stance Part III is forced into.

6.2 Solidification as the analytic completion

At the level of invariants there is a candidate for R_∞ that is already rigorous as a functor, if not yet known to model coarse-graining: Aoki's solidification. Part II installs the identification $K_{\mathrm{op}}(A) \simeq \mathrm{Solid}(K_{\mathrm{alg}}(\underline{A}))$ [2]: the operator K-theory that carries topological-insulator invariants is the *solidification* of the algebraic K-theory of the condensed observable algebra. Solidification is a completion: it inverts the analytic / topological data that algebraic K-theory does not see, and this is exactly the role "passing to the EFT" plays for invariants: discard the lattice-scale algebraic information, keep the topological class.

Conjecture (**IV-4**: solidification commutes with group completion)**.** *Let $M_{d,G}$ be the condensed commutative monoid of stacking classes and K its group completion (Theorem 4.2) computed internally to condensed abelian groups. Then solidification commutes with group completion,*

$$\mathrm{Solid}\big(K(M_{d,G})\big) \;\simeq\; K\big(\mathrm{Solid}(M_{d,G})\big),$$

naturally in (d, G). Equivalently, the analytic completion (solidification) and the algebraic completion (group completion / adding formal inverses) are compatible, so that the invertible solid spectrum may be built in either order.

Conjecture IV-4 is the compatibility that makes "$\mathbf{IP}^{\mathrm{cond}}$ built from lattice data" and "$\mathbf{IP}^{\mathrm{cond}}$ built from solid K-theory" agree. It is plausible on formal grounds—K is a left adjoint (Theorem 4.2) and solidification is a localization / left adjoint [30], and left adjoints compose—but the two adjunctions live in different categories (condensed monoids versus solid modules) and the interchange is not automatic. We state it as a conjecture rather than dress a non-theorem as a corollary.

6.3 What "equivalence" can honestly mean here

We can now say precisely what the EFT passage delivers at each of the three levels of Section 1.3.

- At **(L0)**, the EFT passage is the idempotent $\mathrm{Phases}_{d,G} \to \mathrm{Phases}^{\times}_{d,G}$ of Conjecture IV-5 followed by the microscopic-to-field-theoretic identification of Conjecture IV-2; "lattice = EFT" is a statement about which *set* of phases one gets.

- At **(L1)**, it is the equivalence of spectra $\mathbf{IP}^{\mathrm{cond}}_{d,G} \simeq \mathbf{I}_{\mathrm{FH}}$ after solidification, requiring Conjectures IV-1 and IV-4 to even phrase the completion.

- At **(L2)**, it is naturality over condensed probes, requiring the condensed refinement of IV-1 in an essential way.

None of these is proved. The honest content of this section is the identification of the missing pieces and the fact that the analytic completion step is not mysterious: it is solidification, a functor we already have.

7 The comparison conjecture

We reach the organizing statement. On one side is $\mathbf{IP}^{\mathrm{cond}}_{d,G}$, built from lattice systems; on the other is the Freed–Hopkins classification of invertible field theories, built from bordism and Anderson duality. The comparison conjecture is that they agree.

7.1 The field-theory target

Definition 7.1 (Freed–Hopkins target, [11])**.** Fix a symmetry type H_d (a stable tangential structure, e.g. Spin, Spin $\times\, G$, $\mathrm{Pin}^{\pm}$) with Madsen–Tillmann spectrum MTH. Reflection-positive invertible d-dimensional field theories with symmetry type H_d are classified by the abelian group

$$\left[MTH,\ \Sigma^{d+1} I_{\mathbb{Z}}\right],$$

homotopy classes of spectrum maps into a shift of the Anderson dual $I_{\mathbb{Z}}$ of the sphere. Assembling over d gives a spectrum $\mathbf{I}_{\mathrm{FH}}$ whose homotopy groups are these deformation-class groups; its torsion part is the "beyond group cohomology" content [15] and its free part records the integer invariants (Hall conductances, chiral central charges).

The passage from a short-range-entangled state to such a theory is Freed's theorem that an SRE system defines an invertible field theory [10], and the generalized-cohomology / Ω-spectrum viewpoint of [12, 35] is the statement that these deformation classes are themselves the homotopy of a spectrum—the field-theoretic sibling of Theorem 5.1.

7.2 The comparison map and the conjecture

A lattice invertible phase has a low-energy theory; taking its deformation class defines a comparison map. We isolate it as a hypothesis and then conjecture it is an equivalence.

Definition 7.2 (comparison map, conditional)**.** Assume the recognition hypothesis of Theorem 5.1—that $\mathfrak{Phase}_{d,G}$ is a symmetric-monoidal ∞-groupoid, equivalently Conjecture I-1 of [21]—so that the invertible sector assembles into the spectrum $\mathbf{IP}^{\mathrm{cond}}_{d,G}$; and assume Freed's short-range-entanglement assignment extends naturally in d and in the symmetry data, a naturality we do not construct here. Under these hypotheses one obtains a *comparison map* of spectra

$$c_{d,G} : \ \mathbf{IP}^{\mathrm{cond}}_{d,G} \ \longrightarrow \ \mathbf{I}_{\mathrm{FH}},$$

induced on invertible sectors by sending an SRE lattice family to the deformation class of its low-energy invertible field theory (Freed's assignment [10]). Absent the spectrum-level hypotheses, only the induced map on components $\pi_0(c_{d,G}) : \mathrm{Phases}^{\times}_{d,G} \to \pi_0\mathbf{I}_{\mathrm{FH}}$ is unconditional; it is this π_0 map that the (L0) evidence of Section 8 constrains.

Conjecture (**IV-2**: lattice–EFT comparison equivalence)**.** *Under short-range-entanglement (EFT) hypotheses, the comparison map $c_{d,G}$ is an equivalence after an appropriate completion. Explicitly, at the three levels of Section 1.3:*

- ***(L0)*** $\pi_0(c_{d,G}) : \mathrm{Phases}^{\times}_{d,G} \xrightarrow{\cong} \pi_0\mathbf{I}_{\mathrm{FH}}$ *is an isomorphism of abelian groups: the microscopic invariant equals the deformation class of the associated invertible TQFT;*
- ***(L1)*** $c_{d,G}$ *is an equivalence of connective spectra after solidification / on the relevant completion;*
- ***(L2)*** $c_{d,G}$ *upgrades to a natural equivalence of condensed spectra over profinite probes, compatible with disorder hulls Ω.*

In particular the SSH chain matches its Dirac EFT (Example 9.2).

This is the paper's central claim and it is *open.* The remainder of the paper is evidence and worked cases, never a proof. It is worth being explicit about the two ways IV-2 could fail even for SRE systems: the comparison could be injective but not surjective (some field theory not realized by any lattice model—this is precisely the realizability question handed to Part V [22], where the Kapustin–Fidkowski obstruction [16] shows commuting-projector models cannot realize chiral classes), or surjective but not injective (distinct lattice phases with the same EFT—expected to be prevented by stabilization but not proven in general).

7.3 Structure of the comparison

The comparison sits in a square that summarizes the paper:

$$\begin{tikzcd}
\{\text{SRE lattice families}\}/\mathcal{W} \arrow[r, "\text{stabilize}"] \arrow[d, "\text{low-energy}"'] & \mathrm{Phases}^{\times}_{d,G} \arrow[d, "\pi_0(c_{d,G})"] \\
\{\text{invertible TQFTs}\}/\simeq \arrow[r, "\text{deform. class}"'] & \pi_0 \mathbf{I}_{\mathrm{FH}}
\end{tikzcd}$$

The left vertical is Freed's SRE-to-TQFT assignment; the right vertical is the map IV-2 conjectures to be an isomorphism; the horizontals are the two classifications. Commutativity of the square is definitional; the conjecture is that the right vertical is an isomorphism, whence—given the outer maps—the two notions of "phase" coincide.

8 Evidence

Three bodies of rigorous work constrain Conjecture IV-2. In one regime it is a theorem; in another it is realized by classical K-theory; in a third it has a rigorous homotopical target. We present each honestly, marking exactly how far it reaches.

8.1 $d = 1$: completeness makes (L0) a theorem

Theorem 8.1 ($d = 1$ comparison, from [27, 26]). *For one-dimensional quantum spin chains with on-site finite symmetry G, the operator-algebraic index built from the split property is a* complete *invariant of SPT phases, valued in $H^2(G, \mathbb{T})$. Since $H^2(G, \mathbb{T})$ is exactly the field-theoretic (group-cohomology) classification of 1+1-dimensional G-SPTs, the level-(L0) comparison $\pi_0(c_{1,G})$ is a bijection. In this sense Conjecture IV-2 holds at (L0) in $d = 1$.*

Discussion of proof. The completeness of the operator-algebraic index for $d = 1$ on-site finite symmetry is Ogata's theorem [27] (surveyed in [26]); its value group $H^2(G, \mathbb{T})$ coincides with the group-cohomology classification of (1+1)D bosonic SPTs [5], which is the deformation-class group $\pi_0 \mathbf{I}_{\mathrm{FH}}$ in this case. The identification of the two—that the index of a chain equals the class of its EFT—is the matching used in the operator-algebraic literature; we cite rather than reprove it. Note the hypotheses are exactly the EFT hypotheses of IV-2 specialized to $d = 1$: on-site finite symmetry, unique gapped ground state. □

This is the strongest existing anchor: a regime where "lattice = EFT" is a theorem at the level of phase sets. It does not touch (L1) or (L2)—the split-property index is a π_0 statement—which is exactly why we separated the levels. In $d = 2$ Ogata's $H^3(G, \mathbb{T})$-valued index [28, 27] provides a well-defined bulk invariant that matches the expected group-cohomology label; but its *completeness*—that it separates all 2D SPT phases with on-site finite symmetry, as the split-property index does in $d = 1$—is not established. So the (L0) comparison in $d = 2$ is at present partial: a realized invariant, not a settled bijection.

8.2 Free fermions: the tenfold way realizes the comparison

For free-fermion systems the comparison is classical and complete at (L0) in every dimension, because both sides are computed by the same K-theory. This is the Kitaev periodic table [19], whose mathematical content is K-theoretic [33, 29].

Definition 8.2 (the tenfold way)**.** The ten Altland–Zirnbauer symmetry classes [1] split into two complex classes (A, $AIII$) and eight real classes (AI, BDI, D, $DIII$, AII, CII, C, CI), indexed by a symmetry label $s \in \mathbb{Z}/2$ (complex) or $s \in \mathbb{Z}/8$ (real). In spatial dimension d, the group of strong topological invariants of gapped free-fermion Hamiltonians in the class is

$$\mathcal{G}_{\mathbb{C}}(s,d) = d_{\mathbb{C}}\big((s-d) \bmod 2\big), \qquad \mathcal{G}_{\mathbb{R}}(s,d) = d_{\mathbb{R}}\big((s-d) \bmod 8\big),$$

where the complex and real Bott sequences are

$$d_{\mathbb{C}} : \ (\mathbb{Z},\, 0), \qquad d_{\mathbb{R}} : \ (\mathbb{Z},\, \mathbb{Z}/2,\, \mathbb{Z}/2,\, 0,\, 2\mathbb{Z},\, 0,\, 0,\, 0),$$

read cyclically at indices $0, 1, \ldots$. These are the homotopy groups of the K-theory spectra KU and KO up to the standard $2\mathbb{Z}$ labelling of the degree-4 real generator.

Proposition 8.3 (Bott periodicity and the dimension shift)**.** *The tenfold-way invariant groups of Definition 8.2 are Bott periodic: $\mathcal{G}_{\mathbb{C}}(s,d) = \mathcal{G}_{\mathbb{C}}(s,d+2)$ and $\mathcal{G}_{\mathbb{R}}(s,d) = \mathcal{G}_{\mathbb{R}}(s,d+8)$. They are moreover invariant under the simultaneous shift $(s,d) \mapsto (s+1,d+1)$, so the whole table is determined by the single antidiagonal $s-d$; equivalently, raising the symmetry label and the dimension together leaves the phase group unchanged. These periodicities are the free-fermion incarnation of the spectrum structure of Theorem 5.1, with the dimension shift $d \mapsto d+1$ acting as the degree shift (multiplication by the Bott generator).*

Proof. Both groups are defined by reduction of $s-d$ modulo the period (2 or 8), so periodicity in d and the antidiagonal invariance $(s,d) \mapsto (s+1,d+1)$ are immediate from $\mathcal{G}(s,d) = d_{\bullet}\big((s-d) \bmod p\big)$. That the resulting groups are the correct classifying groups of the ten classes—and that the $d \mapsto d+1$ shift is Bott periodicity of KO/KU—is the content of [19, 33]; we use their identification. The periodicities are verified independently by machine in Section 10. □

Table 1 lists the invariants. Every landmark free-fermion phase sits at its expected entry, and the table is the concrete low-dimensional consistency check the program asks for: in each of these boxes the microscopic K-theory invariant equals the invariant of the massive-Dirac EFT, so Conjecture IV-2 holds at (L0) for free fermions throughout. The free-fermion case also gives the cleanest picture of the dimension-reduction functoriality: the antidiagonal invariance of Proposition 8.3 is the statement that adding a symmetry and adding a dimension cancel, which is dimensional reduction in the sense of [29].

8.3 Ω-spectra: a rigorous target for (L1)

Finally, the homotopical anchors. Kubota's Ω-spectrum (Theorem 5.3) and the parametrized foundations of [4] provide, for the first time, a rigorous *target* at level (L1): a spectrum whose homotopy groups are the invertible gapped systems, against which $c_{d,G}$ can be compared as

class (s)	d=0	1	2	3	4	5	6	7
AI (0)	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$
BDI (1)	$\mathbb{Z}/2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}/2$
D (2)	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
$DIII$ (3)	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII (4)	$2\mathbb{Z}$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0	0	0
CII (5)	0	$2\mathbb{Z}$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0	0
C (6)	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0
CI (7)	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$
A (0)	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
$AIII$ (1)	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

Table 1: The tenfold-way table of Definition 8.2, entry $\mathcal{G}(s, d)$. Real classes (top eight) are 8-periodic along each row and constant along antidiagonals $s - d = \text{const}$; complex classes (bottom two) are 2-periodic. Landmark entries: D at d=1 (Kitaev chain, $\mathbb{Z}/2$); D at d=2 (p+ip, $\mathbb{Z}$); AII at d=2, 3 (quantum spin Hall / 3D TI, $\mathbb{Z}/2$); A at d=2 (integer quantum Hall, $\mathbb{Z}$); $AIII$ at d=1 (SSH, $\mathbb{Z}$).

a map of spectra. They do not prove IV-2 at (L1) (that would require identifying Kubota's spectrum with the Freed–Hopkins target, which is itself open), but they turn (L1) from a slogan into a comparison of two specified spectra. This is the sense in which the program is "rigorous at the module level, conjectural at the global level": the objects are real, the identification is conjectural.

9 Lattice invariants, the SSH example, and transitions

The comparison map $c_{d,G}$ is abstract; the invariants that instantiate it are concrete. We record a rigorous lattice invariant, run the SSH example through the whole apparatus, and connect transitions to the spectrum's boundary map.

9.1 A rigorous lattice invariant

Theorem 9.1 (**IV-C**: Hall conductance as a phase invariant, [17, 18])**.** *For a two-dimensional short-range-entangled lattice state, the Hall conductance is locally computable from the state, is an integer multiple of* e^2/h*, and is constant on gapped phases; it is therefore a well-defined homomorphism* $\mathrm{Phases}^{\times}_{2,G} \to \mathbb{Z}$*. For families over a base, the higher Berry class of [18] generalizes it and unifies Hall conductance with the Thouless charge pump [34].*

This is exactly a component of the comparison map: $\mathrm{Phases}^{\times}_{2,G} \to \mathbb{Z} = \pi_0 \mathbf{I}_{\mathrm{FH}}$ in class A (integer quantum Hall), the free part of Definition 7.1, computed microscopically from the lattice state. That it is locally computable and integer-quantized is the rigorous content behind "the lattice invariant equals the EFT invariant" in this box, and it is the generalized-cohomology invariant $\nu \in E^q(U_f)$ of the program's transition calculus. The Kapustin–Sopenko construction is on the honest side of the Kapustin–Fidkowski wall [16]: it computes the Hall conductance of genuinely chiral states, which no commuting-projector model realizes—a distinction that becomes Part V's subject.

9.2 The SSH chain and its EFT

Example 9.2 (SSH)**.** The Su–Schrieffer–Heeger chain [32] has Bloch Hamiltonian

$$H(k; t_1, t_2) = (t_1 + t_2 \cos k)\,\sigma_x + (t_2 \sin k)\,\sigma_y, \qquad q(k) = t_1 + t_2 e^{ik},$$

with spectrum $E_\pm(k) = \pm|q(k)|$, gapped iff $|t_1| \neq |t_2|$. It is a class $AIII$ system in $d = 1$, whose table entry (Table 1) is $\mathbb{Z}$, the winding number of $q : S^1 \to \mathbb{C}^\times$: $\nu = 1$ for $|t_1| < |t_2|$ and $\nu = 0$ for $|t_1| > |t_2|$. The low-energy theory near the gap-closing at $|t_1| = |t_2|$, $k = \pi$ is a massive 1+1D Dirac fermion, and the winding number is the sign of the Dirac mass, i.e. the deformation class of the Dirac EFT. Thus $c_{1,AIII}$ sends the SSH phase to its EFT class and the two agree: Conjecture IV-2 at (L0) is verified in this box. In the condensed formulation the parameter torus and the momentum circle are replaced by their condensations $\underline{B}, \underline{S^1}$ and the winding becomes a value of the family invariant, but the number is unchanged.

9.3 Transitions and the spectral boundary map

A path in the parameter space B that changes the phase label must cross the gapless discriminant Σ_f. The program organizes this by a relative class: for a family $f : \underline{B} \to \mathfrak{Ham}_{d,G}$ with gapped locus $U_f = \underline{B} \times_{\mathfrak{Ham}_{d,G}} \mathfrak{Gap}_{d,G}$ and label $\nu_f : U_f \to \pi_0 \mathfrak{Phase}_{d,G}$, the obstruction to extending ν across $\Sigma_f = \underline{B} \setminus U_f$ is the relative transition charge $\partial\nu \in E^{q+1}(B, U_f)$. We use the boxed slogan verbatim:

> *A topological transition is crossing the gapless discriminant in the Hamiltonian moduli stack.*

Conjecture (**IV-3**: relative charge = spectral boundary map)**.** *When the invariant ν is valued in the (condensed) generalized cohomology represented by $\mathbf{IP}^{\mathrm{cond}}_{d,G}$, the relative transition charge $\partial\nu \in E^{q+1}(B, U_f)$ is the image of ν_f under the connecting homomorphism of the cofiber sequence*

$$\mathbf{IP}^{\mathrm{cond}}_{d,G}\text{-cohomology of } U_f \;\longrightarrow\; E^{q+1}(B, U_f)$$

associated to the pair (B, U_f); that is, the jump of the phase label across Σ_f is computed by the boundary map of $\mathbf{IP}^{\mathrm{cond}}_{d,G}$. For the SSH chain (Example 9.2) this recovers the statement that the winding number jumps by ± 1 across $|t_1| = |t_2|$, the linking charge of the gapless point.

Conjecture IV-3 is what makes the spectrum $\mathbf{IP}^{\mathrm{cond}}$ do work beyond classification: it turns the transition calculus into a long-exact-sequence computation. Its low-dimensional shadow—the SSH winding jump, the Dirac node as a source of charge—is classical; the conjecture is that this is systematically the boundary map of $\mathbf{IP}^{\mathrm{cond}}$ in every dimension and generalized theory.

10 Formal verification

The algebraic backbone of the paper (the commutative-monoid structure, its group completion, and the Bott periodicity of the tenfold-way table) is elementary enough to be checked by machine, and we do so. The accompanying Haskell package `src/lattice-eft-equivalence/` contains four modules.

`Monoid.hs`: finitely presented commutative monoids and group completion. A finitely presented commutative monoid is represented by generators and relations; elements are normal forms (multisets of generators modulo the relations, reduced by a confluent rewriting of the given relations). The Grothendieck group completion of Definition 4.1 is implemented directly: $K(M)$ is presented as formal differences $[(a, b)]$ with equality decided by the stable-equality test of Proposition 4.3—search for a witness e with $a+b'+e = a'+b+e$ over the (finite, for our examples) reachable set. The universal property (Theorem 4.2(iii)) is exercised by constructing the induced homomorphism $\bar{f}$ from a sample $f : M \to A$ and checking $\bar{f} \circ \gamma = f$.

`TenfoldWay.hs`: the periodic table. The invariant groups $\mathcal{G}_{\mathbb{C}}$ and $\mathcal{G}_{\mathbb{R}}$ of Definition 8.2 are encoded from the Bott sequences $d_{\mathbb{C}}, d_{\mathbb{R}}$ via the antidiagonal formula, and Table 1 is regenerated from the code.

`Main.hs`: demonstrations. Runs group completion on several monoids—the free monoid $\mathbb{Z}_{\geq 0}$ (completing to $\mathbb{Z}$), a product $\mathbb{Z}^2_{\geq 0}$ (completing to $\mathbb{Z}^2$), and the non-cancellative toy $M = \langle a, t \mid t + a = t\rangle$ of Example 4.5, where it exhibits $[a] = 0$ while the multiples $[nt]$ stay distinct, so $K(M) \cong \mathbb{Z}$—then prints Table 1 and exits with status 0.

`Properties.hs`: QuickCheck. The properties tested correspond one-to-one to the paper's claims:

- commutativity, associativity, and unit laws of $\boxtimes$ on monoid elements (Proposition 3.1);
- that $K(M)$ is an abelian group (inverses and associativity) and that γ_M is injective exactly on the cancellative quotient (Proposition 4.3);
- the universal property: for random $f : M \to A$, the induced $\bar{f}$ is a homomorphism and factors f (Theorem 4.2);
- Bott periodicity: $\mathcal{G}_{\mathbb{C}}(s, d) = \mathcal{G}_{\mathbb{C}}(s, d + 2)$, $\mathcal{G}_{\mathbb{R}}(s, d) = \mathcal{G}_{\mathbb{R}}(s, d + 8)$, and the antidiagonal invariance $\mathcal{G}(s, d) = \mathcal{G}(s + 1, d + 1)$ (Proposition 8.3), together with spot checks of the landmark entries of Table 1.

The code is verification, not proof: it checks the finite and periodic content of the elementary claims and guards against off-by-one errors in the table. The conjectures IV-1 through IV-5 are, by their nature, outside its reach.

Code availability. The Haskell package is available at github.com/YonedaAI/topological-phases-of-matter, under `src/lattice-eft-equivalence/`. It builds with GHC and its QuickCheck suite runs the properties listed above.

11 Discussion

What is honestly established. The symmetric-monoidal structure on π_0 and its group completion (Propositions 3.1 and 4.3 and theorem 4.2) are theorems; the reading of ancilla stabilization as the group-completion stabilization element (Remark 4.4) is, we think, the clarifying point of the paper, and it is elementary. The recognition of the invertible sector as a connective spectrum (Theorem 5.1) is a theorem *conditional* on $\mathfrak{Phase}_{d,G}$ being a symmetric-monoidal ∞-groupoid, and its output has a rigorous carrier in Kubota's Ω-spectrum (Theorem 5.3). The free-fermion table and its periodicities (Proposition 8.3) are classical and machine-checked. The $d = 1$ comparison (Theorem 8.1) is a genuine theorem at level (L0).

What is conjectural, and why we did not hide it. The five conjectures IV-1 through IV-5 are the substance of the program at this floor, and none is a theorem. The temptation in a paper like this is to state IV-2 as a "theorem under assumptions" and bury the assumptions; we have instead separated the three levels of equivalence (Section 1.3) precisely so that the true logical status is visible: (L0) is a theorem in two regimes, (L1) has a target but no identification, (L2) is not even statable without the condensed refinement IV-1.

The EFT hypotheses are real. Every positive statement is under short-range entanglement. A lattice system is not automatically an invertible field theory; the assertion that it flows to one is the SRE hypothesis, and it fails for intrinsic topological order. The Freed–Hopkins classification is proved *under* reflection positivity and the EFT axioms [11]; we inherit those hypotheses and do not claim more.

Stabilization collapses distinctions, on purpose. Proposition 4.3 and Example 4.5 show that group completion (and the $(-)^{\mathrm{st}}$ built into $\mathfrak{Phase}_{d,G}$) can send a nonzero phase to zero once enough ancillas are added. This is intended: it is what makes "phase" a stable notion, but it means $\mathbf{IP}^{\mathrm{cond}}$ sees only stable, invertible data. Unstable or fragile distinctions are invisible to it by construction.

Noninvertible order is out of scope. The whole paper lives in the invertible sector. Intrinsic topological order (anyons, modular tensor categories in 2+1D) has no inverse under $\boxtimes$ and no place in a spectrum; a condensed treatment of it would need a condensed *higher* stack of phases and defects, not a K-theory spectrum, and is deferred entirely ([24], global pitfalls). The De Nittis(–Rendel) state-space programme [8, 9] and the moduli-space work of Hsin–Wang [14] are the nearest ordinary-topology precedents; our contribution over them is organizational: placing the same invariants inside a condensed environment where families, disorder, and completions cohabit, not a new invariant, as the program concedes throughout.

Undecidability bounds the renormalization story. Conjecture IV-5 is asserted only on the SRE stratum and only as an existence statement, because the undecidability of the spectral gap [7] forbids any general procedure certifying that a coarse-graining flows to a gapped fixed point. The same wall constrains Part III; the program does not pretend to see past it.

12 Conclusion

We have built the top floor of the condensed tower for topological phases and stated precisely what it would take for the microscopic and field-theoretic classifications to coincide. The provable content is the algebra of stacking: π_0 is a commutative monoid, its group completion is governed by a universal property, and the stabilization element of that completion is exactly ancilla stabilization. The invertible sector is a connective spectrum $\mathbf{IP}^{\mathrm{cond}}_{d,G}$ with a rigorous carrier in Kubota's Ω-spectrum. Against the Freed–Hopkins bordism target we posed the comparison conjecture IV-2 at three levels of strength and marshalled the evidence that pins it down where it can be pinned: Ogata's completeness makes the π_0 comparison a theorem in $d = 1$, and the tenfold-way K-theory, with its Bott periodicities, which we verified by machine, realizes it for free fermions in every dimension. The remaining conjectures locate the missing analysis: solidification as the EFT completion (IV-4, IV-5), the condensed refinement of the spectrum (IV-1), and the transition charge as a spectral boundary map (IV-3).

The honest summary is that the equivalence between lattice models and effective field theories is, at π_0 and under EFT hypotheses, either a theorem (Ogata, free fermions) or a well-posed and evidence-constrained conjecture, and that its spectrum- and family-level forms are exactly what the condensed refinement is for. Part V takes the baton with the realizability question (which of these classes are built by actual lattices), and Part VI assembles the tower into the program's Master Conjecture, of which our IV-2 is the top.

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