Positivity, C^*-Norms, and Condensed State Spaces of Quasi-Local Algebras

Part II of *Topological Phases of Matter in the Condensed-Mathematics Paradigm*

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Abstract

The observable side of a quantum lattice system is a quasi-local C^*-algebra; its states, its positivity structure, and its K-theory are the raw material from which topological invariants of phases are cut. This paper recasts that material in condensed mathematics, separating what is provable today from what is not. For a spin system with finite on-site dimension the quasi-local algebra $\mathcal{A}$ is a separable unital approximately finite-dimensional (AF) algebra, uniformly hyperfinite (UHF) in the homogeneous case, and its condensation $\underline{\mathcal{A}}(S) = C(S, \mathcal{A})$ is a sheaf of C^*-algebras on the site of profinite sets, *light* because $\mathcal{A}$ is separable. We prove that the weak-$*$ compact convex state space $\mathcal{S}(\mathcal{A})$ condenses to a compact Hausdorff condensed set on which the condensation functor is fully faithful, and that positivity together with normalization cuts it out as a closed condensed subobject of the condensed dual ball (Theorem II-A). The Gelfand–Naimark–Segal construction is a functor on pointed C^*-algebras, and on a uniformly gapped family the ground-state section is weak-$*$ continuous by the Bachmann–Michalakis–Nachtergaele–Sims spectral-flow cocycle, so it is a genuine point of the condensed state space over the gapped locus, the interface with Parts I and III. The load-bearing bridge is Aoki's solidification theorem: solidification of the connective algebraic K-theory of the condensed algebra recovers connective semitopological K-theory, and, for a real Banach algebra, the full operator K-theory after inverting the Bott class (Theorem II-B). The tenfold-way invariants of free-fermion and disordered phases are then values of an intrinsically condensed-mathematical functor, though the numbers themselves are unchanged. For disorder we recall Bellissard's crossed product $C(\Omega) \rtimes \mathbb{Z}^d$ over a profinite hull $\Omega = Q^{\mathbb{Z}^d}$ and show that its condensation is functorial in the finite quotients of Ω (Theorem II-C), so profinite probes *match* the physical configuration space. Five numbered conjectures record what remains open. Accompanying Haskell code verifies positivity, the GNS construction, and the qubit Bloch ball.

Contents

1 Introduction

1.1 The problem this paper addresses

The program that this series develops treats a topological phase of matter as a connected component of a stabilized condensed stack of uniformly gapped local Hamiltonians. The prospectus for the program lists five analytic inputs that the condensed formalism organizes but does not by itself supply: locality and Lieb–Robinson estimates; positivity and C^*-norm conditions; existence and stability of a thermodynamic gap; the passage from lattice models to effective field theories; and the physical realizability of abstract classes. This is the paper for the second input.

The word "positivity" is doing real work. A quantum lattice system is presented combinatorially (sites, local Hilbert spaces, a decaying interaction), but its physical content is carried by an algebra of observables and by the states on that algebra. That algebra is not an arbitrary Banach algebra: it is a C^*-algebra, and the extra axiom $\|a^*a\| = \|a\|^2$ forces a positivity structure that determines its state space, its representation theory, and ultimately its K-theory. Topological invariants of free-fermion and disordered phases are, in the end, classes in the operator K-theory of this algebra. If the condensed reformulation is to reach those invariants, it must first carry the C^*-algebra, its states, and its positivity into the condensed world without loss. Our task is to check that this transfer is faithful, to identify what becomes provable once it is done, and to name the bridge that turns the operator-K invariant into an intrinsically condensed object.

1.2 Results and stance

We hold to the editorial discipline of the series: a statement is labelled *Theorem* or *Proposition* only when it has a complete proof assembled from cited, present-day mathematics, and everything else is a numbered *Conjecture*. Under that discipline the paper proves the following.

- The quasi-local algebra of a finite-on-site-dimension spin system is a separable unital AF (UHF in the homogeneous case) C^*-algebra (Proposition 2.2), and its condensation $\underline{\mathcal{A}}$ is a sheaf of C^*-algebras on profinite sets, light because $\mathcal{A}$ is separable (Proposition 2.3).
- The state space $\mathcal{S}(\mathcal{A})$ is weak-$*$ compact convex Hausdorff, its condensation is a compact Hausdorff condensed set on which condensation is fully faithful, and positivity plus normalization exhibit it as a closed condensed subobject of the condensed dual ball (Theorem II-A, labelled II-A in the program).
- GNS is a functor on the category of C^*-algebras with a state (Proposition 4.2), and on a uniformly gapped C^1-family the ground-state section is weak-$*$ continuous (Theorem 4.3), hence a point of the condensed state space over the gapped locus.
- Aoki's solidification theorem identifies operator K-theory of a real Banach algebra with the solidification of the algebraic K-theory of its condensation (Theorem II-B, labelled II-B); consequently the tenfold-way invariants factor through a condensed-mathematical functor (Corollary 6.1), a statement we are careful not to inflate.
- For a profinite disorder hull the covariant observable algebra is Bellissard's crossed product $C(\Omega) \rtimes \mathbb{Z}^d$, a separable C^*-algebra whose K-theory carries the disorder-averaged invariants, and its condensation is functorial in the finite quotients of Ω (Theorem II-C, labelled II-C).

The open content is organized as Conjectures II-1 to II-5: that positivity and the C^*-identity cut the Hamiltonian stack out as a closed condensed substack; that the solid invariant is natural in the disorder hull; that a Real/KKO refinement recovers the eightfold KO-degrees of the Kitaev table; that the condensed state space is a classifying object for condensed representations; and that the split property and superselection structure admit condensed enhancements.

A word on what this paper does *not* claim. It does not produce a new physical invariant. The Chern number of a Chern insulator, the $\mathbb{Z}_2$ index of a time-reversal-invariant insulator, and the winding number of the SSH chain are unchanged. What changes is their provenance: they are exhibited as values of a functor whose inputs and intermediate objects are condensed. The value proposition is organizational, and we say so plainly rather than dress it as a breakthrough.

1.3 Relation to companion papers

This is Part II of six. Part I, *Condensed Locality* [23], builds the interaction Banach space $\mathcal{B}_F$ and shows that the Heisenberg dynamics is a morphism of light condensed sets $\underline{\mathbb{R}} \times \underline{\mathcal{B}_F} \to \underline{\mathrm{Aut}(\mathcal{A})}$; that action is by automorphisms of the very algebra whose states we condense here, and the equivalence class $\mathcal{W}$ that Part I generates through quasi-adiabatic continuation is what makes the ground-state section of Theorem 4.3 well defined up to the physically correct notion of sameness. Part III, *The Uniformly Gapped Substack* [24], is where the uniform-gap hypothesis of Theorem 4.3 is analyzed; our continuous ground-state section is a section over their gapped locus $U_f \subset \mathfrak{Gap}_{d,G}$, and the weak-$*$ continuity we prove is the reason π_0 of their localized stack is well posed on the observable side. Part IV, *From Lattice Models to Effective Field Theories* [25], group-completes the invertible sector into the spectrum $\mathbf{IP}^{\mathrm{cond}}_{d,G}$; the Aoki bridge of Theorem II-B is precisely the analytic completion their construction requires, and our Corollary 6.1 supplies the observable-algebra K-theory they compare against Freed–Hopkins bordism. Part V, *Physical Realizability* [26], studies which of the operator-K classes we place in condensed form are realized by gapped lattice models; our disorder functoriality (Theorem II-C, Conjecture II-2) is the input to their functorial-realizability question over disorder hulls. Part VI, *A Modular Research Program* [27], assembles the modules; this paper is module 2, the positivity/C^* layer, and its emergent contribution to the synthesis is the functorial solid invariant of disordered families.

1.4 Notation

We use the canonical macros of the series: $\underline{X}$ for condensation with $\underline{X}(S) = \mathrm{Cont}(S, X)$, $\mathfrak{Ham}_{d,G}$, $\mathfrak{Gap}_{d,G}$, $\mathfrak{Phase}_{d,G}$ for the moduli stacks, $\mathcal{W}$ for the inverted equivalences, K_{op} and K_{alg} for operator and algebraic K-theory, Solid for solidification, and $\Omega = Q^{\mathbb{Z}^d}$ for the profinite disorder hull. Paper-local objects are $\mathcal{A}$ (the quasi-local algebra), $\mathcal{S}(\mathcal{A})$ and $\mathcal{P}(\mathcal{A})$ (states and pure states), and $\mathfrak{A}$ (the disordered observable algebra). Throughout, C^*-algebras are complex and unital unless we say otherwise; the sole exception is Section 6, where Aoki's theorem is a statement about *real* algebras and we flag the change explicitly.

2 Condensed mathematics and quasi-local C^*-algebras

We recall only what we use, and we fix conventions so that the later sections read cleanly. Nothing in this section is new; the point is to make the two categories, condensed sets and C^*-algebras, interoperate.

2.1 Condensed sets and profinite probes

A *profinite set* is a cofiltered limit of finite sets, equivalently a compact, Hausdorff, totally disconnected topological space. The category of profinite sets carries the topology whose covers are the finite jointly surjective families; a *condensed set* is a sheaf of sets for this topology [1, 4]. The basic examples come from spaces.

Definition 2.1 (Condensation)**.** For a topological space X, its *condensation* $\underline{X}$ is the condensed set

$$\underline{X}(S) \;=\; \mathrm{Cont}(S, X), \qquad S \text{ profinite},$$

with restriction along maps of profinite sets given by precomposition.

The assignment $X \mapsto \underline{X}$ is functorial. On the subcategory of compactly generated (weak Hausdorff) spaces it is fully faithful, and it preserves finite products, because $\mathrm{Cont}(S, X \times Y) = \mathrm{Cont}(S, X) \times \mathrm{Cont}(S, Y)$ for every probe S [1]. Full faithfulness is the technical reason a condensed reformulation loses nothing: an ordinary compact space and its condensation carry the same maps into and out of them.

For the disorder applications we will want a size-controlled variant. A profinite set is *light* if it is second countable, equivalently a countable cofiltered limit of finite sets; *light condensed sets* are sheaves on the light profinite sets [3]. The condensation of a separable (metrizable, second countable) space is determined by its values on light probes, and it is convenient to regard it as a light condensed set. Our algebras are separable, so light condensed structure suffices everywhere below, and we avoid the set-theoretic subtleties of the unrestricted theory.

2.2 Quasi-local algebras of spin systems

Fix a countable set L of sites (for definiteness a lattice $\mathbb{Z}^d$ with the graph metric, though only countability and the net of finite subsets matter here). To each site x attach a finite-dimensional Hilbert space $\mathcal{H}_x$ with $\dim \mathcal{H}_x = n_x < \infty$, and set $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x$ for finite $\Lambda \Subset L$. The *local algebra* at Λ is the full matrix algebra

$$\mathcal{A}_\Lambda \;=\; \mathcal{B}(\mathcal{H}_\Lambda) \;\cong\; \bigotimes_{x\in\Lambda} M_{n_x}(\mathbb{C}).$$

For $\Lambda \subseteq \Lambda'$ the map $a \mapsto a \otimes \mathbf{1}_{\Lambda'\setminus\Lambda}$ is a unital $*$-monomorphism $\mathcal{A}_\Lambda \hookrightarrow \mathcal{A}_{\Lambda'}$. The *local subalgebra* is the union $\mathcal{A}_{\mathrm{loc}} = \bigcup_{\Lambda\Subset L} \mathcal{A}_\Lambda$, an incomplete normed $*$-algebra, and the *quasi-local algebra* is its completion

$$\mathcal{A} \;=\; \overline{\mathcal{A}_{\mathrm{loc}}}^{\,\|\cdot\|},$$

the C^*-inductive limit of the net $(\mathcal{A}_\Lambda)_\Lambda$.

Proposition 2.2 (UHF/AF structure of the quasi-local algebra)**.** *The quasi-local algebra $\mathcal{A}$ is a separable unital C^*-algebra. It is approximately finite-dimensional: an inductive limit of finite-dimensional C^*-algebras with unital connecting maps. If L is infinite and $n_x = n$ for all x, then $\mathcal{A}$ is the uniformly hyperfinite algebra of type n^∞,*

$$\mathcal{A} \;\cong\; \overline{\bigotimes_{x\in L} M_n(\mathbb{C})} \;=\; M_n^{\otimes\infty}.$$

Proof. Enumerate $L = \{x_1, x_2, \dots\}$ and set $\Lambda_k = \{x_1, \dots, x_k\}$. The finite subsets Λ_k are cofinal in the net of all finite subsets, so $\mathcal{A} = \overline{\bigcup_k \mathcal{A}_{\Lambda_k}}$ is the closure of an increasing union of the finite-dimensional C^*-algebras $\mathcal{A}_{\Lambda_k}$ along the unital inclusions $\mathcal{A}_{\Lambda_k} \hookrightarrow \mathcal{A}_{\Lambda_{k+1}}$. By definition this exhibits $\mathcal{A}$ as the C^*-inductive limit $\varinjlim_k \mathcal{A}_{\Lambda_k}$, which is AF. Each $\mathcal{A}_{\Lambda_k}$ is finite dimensional, hence separable, and it contains the countable $\mathbb{Q}(i)$-$*$-subalgebra of matrices with Gaussian-rational entries; the union of these over k is a countable norm-dense $*$-subalgebra of $\mathcal{A}$, so $\mathcal{A}$ is separable. The unit of $\mathcal{A}_{\Lambda_1}$ is a unit for the limit. When $n_x = n$ for all x, each $\mathcal{A}_{\Lambda_k} \cong M_n(\mathbb{C})^{\otimes k} \cong M_{n^k}(\mathbb{C})$ and the connecting maps are the standard unital embeddings $M_{n^k} \hookrightarrow M_{n^{k+1}}$, $a \mapsto a \otimes \mathbf{1}_n$; the inductive limit of this system is by definition the UHF algebra of type n^∞. □

The operator-algebraic viewpoint on gapped phases takes exactly this $\mathcal{A}$ as the algebra of quasi-local observables; the ground states of physical interest are states on $\mathcal{A}$ [19, 9]. That $\mathcal{A}$ is AF, and in particular nuclear and separable, is what will make its condensation light and its state space metrizable.

2.3 Condensation of a C^*-algebra

A C^*-algebra is an algebra object with involution in the symmetric monoidal category of (compactly generated) topological spaces: multiplication, addition, scalar action, and involution are continuous, and the norm makes the underlying set a complete metric space. Since condensation preserves finite products, it carries this structure across.

Proposition 2.3 (Condensation is a sheaf of C^*-algebras; separable $\Rightarrow$ light)**.** *Let A be a C^*-algebra. The condensation $\underline{A}$, $\underline{A}(S) = C(S, A)$, is a C^*-algebra object in condensed sets: for every profinite S the value $C(S, A)$ is a C^*-algebra under pointwise operations and the supremum norm, and the restriction maps are $*$-homomorphisms. If A is unital, so is $\underline{A}$. If A is separable, $\underline{A}$ is a light condensed set.*

Proof. That $S \mapsto \mathrm{Cont}(S, A)$ is a sheaf, i.e. a condensed set, is the defining property of the condensation of the topological space A [1]. The C^*-operations of A are continuous maps $A \times A \to A$ and $A \to A$; post-composition sends a continuous S-family to a continuous S-family, and because condensation preserves products these assemble into morphisms of condensed sets $\underline{A}\times\underline{A} \to \underline{A}$ and $\underline{A} \to \underline{A}$ satisfying the algebra and involution axioms levelwise. Concretely, for compact S the space $C(S, A)$ with $(f \cdot g)(s) = f(s)g(s)$, $f^*(s) = f(s)^*$, and $\|f\| = \sup_s \|f(s)\|$ is a C^*-algebra: it is a Banach $*$-algebra, and the C^*-identity is inherited pointwise,

$$\|f^* f\| = \sup_s \|f(s)^* f(s)\| = \sup_s \|f(s)\|^2 = \Big(\sup_s \|f(s)\|\Big)^2 = \|f\|^2.$$

Restriction along $\theta : S' \to S$ is $f \mapsto f \circ \theta$, evidently a unital $*$-homomorphism. When A is unital the constant function at $\mathbf{1}_A$ is a unit for each $C(S, A)$, compatible with restriction. Finally, if A is separable it is second countable, so its condensation is determined by its restriction to light profinite probes and is a light condensed set [3]. $\square$

Remark 2.4. The sheaf condition here is descent for A-valued continuous functions along finite jointly surjective families $\{S_i \to S\}$: such a family gives a topological quotient $\coprod_i S_i \twoheadrightarrow S$ of compact Hausdorff spaces, and a continuous A-valued function on S is the same as a compatible family on the S_i agreeing on the fibered products $S_i \times_S S_j$. This is the reason condensation, and not merely the underlying set, is the right target: it remembers how continuous families glue.

3 Positivity and the condensed state space

We now condense the state space and read off positivity as a closed condition. This is the first place where the C^*-axiom, and not just the Banach structure, is used.

3.1 States, positivity, and weak-$*$ compactness

Let A be a unital C^*-algebra. A *state* is a linear functional $\phi : A \to \mathbb{C}$ that is positive, $\phi(a^*a) \geq 0$ for all a, and normalized, $\phi(\mathbf{1}) = 1$. Positivity forces continuity with $\|\phi\| = \phi(\mathbf{1}) = 1$, so states lie in the unit ball $(A^*)_1$ of the dual. Write $\mathcal{S}(A)$ for the set of states and $\mathcal{P}(A)$ for its extreme points, the *pure states*.

Lemma 3.1 (Weak-$*$ compactness)**.** *$\mathcal{S}(A)$ is a weak-$*$ closed, convex subset of $(A^*)_1$, hence weak-$*$ compact and Hausdorff. If A is separable, $\mathcal{S}(A)$ is metrizable, so it is a compact convex Polish space, and $\mathcal{S}(A) = \overline{\mathrm{conv}}\, \mathcal{P}(A)$.*

Proof. By Banach–Alaoglu $(A^*)_1$ is weak-$*$ compact and Hausdorff. For each fixed $a \in A$ the evaluation $\mathrm{ev}_a : \phi \mapsto \phi(a)$ is weak-$*$ continuous, so the sets $\{\phi : \phi(a^*a) \geq 0\} = \mathrm{ev}_{a^*a}^{-1}([0, \infty))$ and $\{\phi : \phi(\mathbf{1}) = 1\} = \mathrm{ev}_{\mathbf{1}}^{-1}(\{1\})$ are weak-$*$ closed. Their intersection over all a is $\mathcal{S}(A)$, which is therefore weak-$*$ closed in the compact set $(A^*)_1$, hence compact; convexity is immediate from linearity of the conditions. When A is separable the weak-$*$ topology on $(A^*)_1$ is metrizable, so the closed subset $\mathcal{S}(A)$ is a compact metric space; it is convex and compact, and Krein–Milman gives $\mathcal{S}(A) = \overline{\mathrm{conv}}\, \mathcal{P}(A)$. $\square$

Example 3.2 (Matrix algebras and the Bloch ball)**.** For $A = M_n(\mathbb{C})$ every state is $\phi_\rho(a) = \mathrm{Tr}(\rho a)$ for a unique density matrix $\rho \geq 0$ with $\mathrm{Tr}\, \rho = 1$, so $\mathcal{S}(M_n)$ is the set of density matrices and $\mathcal{P}(M_n)$ is the set of rank-one projections, a copy of $\mathbb{C}P^{n-1}$. For $n = 2$, writing $\rho = \frac{1}{2}(\mathbf{1} + \vec{r} \cdot \vec{\sigma})$ with $\vec{\sigma}$ the Pauli matrices, positivity is exactly $|\vec{r}| \leq 1$: the state space $\mathcal{S}(M_2)$ is the closed unit ball $B^3 \subset \mathbb{R}^3$ (the Bloch ball) and $\mathcal{P}(M_2)$ is its boundary sphere $S^2 = \mathbb{C}P^1$. The accompanying Haskell code takes exactly this picture as its concrete testbed.

3.2 The condensed state space

Theorem II-A (Condensed state space)**.** *Let A be a unital C^*-algebra. Then $\underline{\mathcal{S}(A)}$ is a compact Hausdorff condensed set, and the condensation functor is fully faithful on it. Positivity and normalization exhibit $\underline{\mathcal{S}(A)}$ as a closed condensed subobject of the condensed dual ball $\underline{(A^*)_1}$: for every profinite S,*

$$\underline{\mathcal{S}(A)}(S) = \big\{\, f \in C\big(S, (A^*)_1\big) : f(s)(a^*a) \geq 0 \text{ and } f(s)(\mathbf{1}) = 1 \text{ for all } s \in S,\ a \in A \,\big\},$$

a weak-$*$ closed subset of $C(S, (A^*)_1)$. If A is separable the subobject is a closed inclusion of light condensed sets.

Proof. By Lemma 3.1, $\mathcal{S}(A)$ is a compact Hausdorff space, so its condensation is a compact Hausdorff condensed set, and full faithfulness of $\underline{(-)}$ on compact Hausdorff spaces is [1]. For a profinite S a point of $\underline{\mathcal{S}(A)}(S)$ is a continuous map $f : S \to \mathcal{S}(A)$, i.e. a continuous map $f : S \to (A^*)_1$ each of whose values is a state; this is the displayed set. Each defining condition is closed: for fixed a and s the map $f \mapsto f(s)(a^*a)$ is continuous on $C(S, (A^*)_1)$ (composite of evaluation at s and ev_{a^*a}), so $\{f : f(s)(a^*a) \geq 0\}$ is closed, and likewise for normalization; the intersection over $s \in S$ and $a \in A$ is closed. Because the inclusion $\mathcal{S}(A) \hookrightarrow (A^*)_1$ is a closed embedding of compact Hausdorff spaces and condensation preserves closed embeddings of such spaces, the induced map $\underline{\mathcal{S}(A)} \to \underline{(A^*)_1}$ is a closed inclusion of condensed sets. When A is separable both spaces are second countable, so both condensations are light by Proposition 2.3 and the inclusion is one of light condensed sets. ☐

Remark 3.3 (Why closedness is the right statement). A condensed reformulation of "the state space" has to answer: what is a *family* of states over a parameter, and when does a limit of states stay a state? Theorem II-A answers both at once. A family over S is a point of $\underline{\mathcal{S}(A)}(S)$; positivity is preserved under the profinite probes because it is a closed condition, so a profinite limit of states is a state. This is the condensed avatar of the elementary fact that the state space is weak-$*$ closed, upgraded to a statement about all profinite families simultaneously.

3.3 Positivity as a condensed order structure

It is worth saying what "positivity is a closed condition" means at the level of the algebra, not just the dual, because that is where the C^*-identity enters and because it is the form the condition takes when one tries to characterize $\mathfrak{Ham}_{d,G}$ intrinsically (Conjecture II-1). The positive cone of a C^*-algebra A is

$$A_+ \;=\; \{\, a^*a : a \in A \,\} \;=\; \{\, a \in A_{\mathrm{sa}} : \sigma(a) \subseteq [0, \infty) \,\},$$

a closed convex cone in the real Banach space A_{sa} of self-adjoint elements, with $A_+ \cap (-A_+) = \{0\}$; together with the order unit $\mathbf{1}$ it makes $(A_{\mathrm{sa}}, A_+, \mathbf{1})$ an order-unit space whose order-unit state space is exactly $\mathcal{S}(A)$.

Proposition 3.4 (Condensation preserves the positive cone and order unit)**.** *Let A be a unital C^*-algebra. The condensation $\underline{A_+}$ is a closed sub-condensed-set of $\underline{A_{\mathrm{sa}}}$, and for every profinite S,*

$$\underline{A_+}(S) \;=\; \{\, f \in C(S, A_{\mathrm{sa}}) : f(s) \in A_+ \text{ for all } s \in S \,\} \;=\; C(S, A_+).$$

The order unit condenses to the constant section $\mathbf{1}$, and a state on A is the same as a morphism of condensed sets $\underline{A_{\mathrm{sa}}} \to \underline{\mathbb{R}}$ that is $\mathbb{R}$-linear on probes, carries $\underline{A_+}$ into $\underline{\mathbb{R}_{\geq 0}}$, and sends $\mathbf{1}$ to 1.

Proof. The cone A_+ is a closed subset of A_{sa}, so a continuous map $f : S \to A_{\mathrm{sa}}$ takes values in A_+ if and only if $f(s) \in A_+$ for every s; this is the displayed equality $\underline{A_+}(S) = C(S, A_+)$, and closedness of the inclusion into $\underline{A_{\mathrm{sa}}}$ follows as in Theorem II-A. That condensation carries the order unit to the constant section is immediate. For the last clause, a probe-wise $\mathbb{R}$-linear morphism $\underline{A_{\mathrm{sa}}} \to \underline{\mathbb{R}}$ is, by full faithfulness of condensation on the compactly generated space A_{sa}, a continuous $\mathbb{R}$-linear functional $\phi : A_{\mathrm{sa}} \to \mathbb{R}$; the conditions $\phi(A_+) \subseteq \mathbb{R}_{\geq 0}$ and $\phi(\mathbf{1}) = 1$ are precisely those making ϕ extend to a state on A. □

This is the internal counterpart of Theorem II-A: positivity lives on the algebra as a closed condensed cone, and a state is a cone- and unit-preserving condensed functional. The C^*-identity is what pins this cone to the spectrum, and it is the relation Conjecture II-1 would use to cut $\mathfrak{Ham}_{d,G}$ out of a formal-interaction stack.

4 The GNS construction as a functor

The link between a state and a Hilbert-space representation is the Gelfand–Naimark–Segal construction. We record its functoriality (elementary, but what lets it interact with the condensed structure) and then prove the continuity statement that matters for phases: on a uniformly gapped family, the ground-state section into $\mathcal{S}(\mathcal{A})$ is weak-$*$ continuous.

4.1 Functoriality of GNS

Definition 4.1 (The category of pointed C^*-algebras)**.** Let $\mathsf{C^*St}$ be the category whose objects are pairs (A, ϕ) of a unital C^*-algebra and a state, and whose morphisms $(A, \phi) \to (B, \psi)$ are unital $*$-homomorphisms $\theta : A \to B$ with $\psi \circ \theta = \phi$ (state-preserving maps).

Given (A, ϕ), the sesquilinear form $\langle a, b\rangle_\phi = \phi(a^* b)$ is positive semidefinite; its kernel $N_\phi = \{a : \phi(a^* a) = 0\}$ is a closed left ideal (by the Cauchy–Schwarz inequality $|\phi(a^* b)|^2 \leq \phi(a^* a)\phi(b^* b)$); the completion $\mathcal{H}_\phi = \overline{A/N_\phi}$ is a Hilbert space; left multiplication descends to a $*$-representation $\pi_\phi(a)[b] = [ab]$; and the class $\Omega_\phi = [\mathbf{1}]$ is a cyclic unit vector with $\phi(a) = \langle \Omega_\phi, \pi_\phi(a)\Omega_\phi\rangle$.

Proposition 4.2 (GNS is a functor)**.** *The assignment $(A, \phi) \mapsto (\mathcal{H}_\phi, \pi_\phi, \Omega_\phi)$ extends to a functor from $\mathsf{C^*St}$ to the category of pointed cyclic representations with intertwining isometries. A state-preserving map $\theta : (A, \phi) \to (B, \psi)$ induces the isometry*

$$V_\theta : \mathcal{H}_\phi \to \mathcal{H}_\psi, \qquad V_\theta [a]_\phi = [\theta(a)]_\psi,$$

satisfying $V_\theta \Omega_\phi = \Omega_\psi$ and $V_\theta\, \pi_\phi(a) = \pi_\psi(\theta(a))\, V_\theta$, with $V_{\theta' \circ \theta} = V_{\theta'} V_\theta$ and $V_{\mathrm{id}} = \mathrm{id}$.

Proof. Because $\psi \circ \theta = \phi$ we have $\langle \theta(a), \theta(b) \rangle_\psi = \psi(\theta(a)^*\theta(b)) = \psi(\theta(a^*b)) = \phi(a^*b) = \langle a, b\rangle_\phi$, so θ maps N_ϕ into N_ψ and the map $[a]_\phi \mapsto [\theta(a)]_\psi$ is a well-defined isometry on the dense subspaces, extending to an isometry V_θ of the completions. Evaluating on the unit gives $V_\theta \Omega_\phi = \Omega_\psi$, and for $a, b \in A$, $V_\theta \pi_\phi(a)[b]_\phi = V_\theta[ab]_\phi = [\theta(a)\theta(b)]_\psi = \pi_\psi(\theta(a))[\theta(b)]_\psi = \pi_\psi(\theta(a))V_\theta[b]_\phi$, i.e. V_θ intertwines. Functoriality in θ is immediate from the definition on representatives. □

4.2 Continuity of the ground-state section on gapped families

The physically decisive continuity statement is not about GNS of a single state but about a *family* of ground states. We use the automorphic-equivalence theorem for gapped phases, which is the interface with Parts I and III.

Theorem 4.3 (Weak-$*$ continuity of the ground-state section)**.** *Let $s \mapsto \Phi_s$, $s \in [0, 1]$, be a family of interactions that is C^1 in the interaction Banach norm $\mathcal{B}_F$ of Part I, and suppose the family is* uniformly gapped*: the thermodynamic-limit dynamics has a unique ground state ω_s on $\mathcal{A}$ separated from the rest of the spectrum by a gap $\geq \Delta > 0$ uniformly in s. Then there is a strongly continuous family of quasi-local automorphisms $\alpha_s \in \mathrm{Aut}(\mathcal{A})$ with $\omega_s = \omega_0 \circ \alpha_s$, and for every local observable $a \in \mathcal{A}_{\mathrm{loc}}$ the map $s \mapsto \omega_s(a)$ is continuous. Consequently $s \mapsto \omega_s$ is a weak-$*$ continuous map $[0, 1] \to \mathcal{S}(\mathcal{A})$; taking a profinite probe $S \to [0, 1]$ landing in the gapped locus, the ground state is a point of $\underline{\mathcal{S}(\mathcal{A})}(S)$.*

Proof. Under the stated hypotheses the Bachmann–Michalakis–Nachtergaele–Sims theorem on automorphic equivalence within a gapped phase produces the quasi-adiabatic spectral-flow cocycle: a strongly continuous family of $*$-automorphisms α_s of $\mathcal{A}$, generated by a time-dependent quasi-local interaction built from $\dot{\Phi}_s$ and the gap, with $\omega_s = \omega_0 \circ \alpha_s$ [6, 7]. The generator is quasi-local with Lieb–Robinson-controlled tails, so for a local observable a the orbit $s \mapsto \alpha_s(a)$ is norm-continuous (indeed C^1) [8, 23]. Hence $s \mapsto \omega_s(a) = \omega_0(\alpha_s(a))$ is continuous, being the composition of a norm-continuous $\mathcal{A}$-valued map with the bounded functional ω_0. Local observables are norm-dense in $\mathcal{A}$ and each ω_s is a state with $\|\omega_s\| = 1$; a uniformly bounded net of functionals that converges on a dense set converges weak-$*$, so continuity on $\mathcal{A}_{\mathrm{loc}}$ upgrades to weak-$*$ continuity on all of $\mathcal{A}$. Thus $s \mapsto \omega_s$ is a continuous map into $\mathcal{S}(\mathcal{A})$ with its weak-$*$ topology. For a profinite S with a continuous map $S \to [0, 1]$ whose image lies in the gapped locus, the composite $S \to \mathcal{S}(\mathcal{A})$ is continuous, i.e. an element of $\underline{\mathcal{S}(\mathcal{A})}(S)$ by Theorem II-A. □

Remark 4.4 (What is used, and what is deferred). The theorem uses the uniform gap as a hypothesis; whether a given family satisfies it is the subject of Part III, and in general the question is undecidable, so we do not attempt a criterion here. The output is exactly the datum the program needs on the observable side: a continuous section of states over the gapped locus, which is what makes the phase label ν_f locally constant and the whole π_0 story well posed. The representation-theoretic refinement (that GNS assembles into a continuous field of Hilbert spaces over the base) is delicate because the GNS representation can jump in dimension, and we record the clean version we can prove (continuity of the vector-state section) rather than overclaim a continuous field. The full statement is part of Conjecture II-4.

5 Free fermions and the real observable algebra

Before invoking Aoki's theorem we make concrete the algebra to which it will be applied. Free-fermion systems are the best-developed setting for topological classification, and they are where the *real* structure that Aoki's theorem requires is visible on the nose. This section fixes the observable algebra, its quasi-free ground states, and the real structure imposed by the antiunitary symmetries, and it identifies the SSH winding number as an operator-K class so that the corollary of Section 6 lands on a familiar object.

5.1 The CAR algebra and quasi-free ground states

For a free-fermion system with single-particle Hilbert space $\mathfrak{h} = \ell^2(L)$ (one orbital per site, for simplicity) the observable algebra is the algebra of canonical anticommutation relations $\mathrm{CAR}(\mathfrak{h})$, the unital C^*-algebra generated by $\{a(v) : v \in \mathfrak{h}\}$ with $v \mapsto a(v)^*$ linear and

$$\{a(v), a(w)^*\} = \langle v, w\rangle \mathbf{1}, \qquad \{a(v), a(w)\} = 0,$$

where the inner product is linear in its second argument (the physics convention), so that $v \mapsto a(v)$ is antilinear and $v \mapsto a(v)^*$ is linear. For separable infinite-dimensional $\mathfrak{h}$ the algebra $\mathrm{CAR}(\mathfrak{h})$ is the UHF algebra of type 2^∞; in particular it is a special case of the quasi-local algebra of Proposition 2.2, and everything in Sections 2 to 4 applies to it verbatim.

A quadratic Hamiltonian is specified by a self-adjoint single-particle operator $h = h^*$ on $\mathfrak{h}$, and when h has a spectral gap at the Fermi level 0 the ground state is the *quasi-free* state ω_P determined by the Fermi projection $P = \chi_{(-\infty,0)}(h)$, through the two-point function

$$\omega_P\big(a(v)^* a(w)\big) = \langle w, Pv\rangle,$$

with Wick's theorem fixing all higher correlators. Two facts connect this to the earlier sections. First, ω_P is a state on $\mathrm{CAR}(\mathfrak{h})$, hence a point of $\mathcal{S}(\mathrm{CAR}(\mathfrak{h}))$, and the map $P \mapsto \omega_P$ is weak-$*$ continuous in P; a uniformly gapped family of single-particle Hamiltonians thus gives a continuous ground-state section exactly as in Theorem 4.3, and one does not even need the full automorphic-equivalence machinery in the quasi-free case because the section is visibly continuous in the Fermi projection. Second, the spectral gap forces P to have exponentially decaying off-diagonal, so P is a *controlled* (quasi-local) projection — the free-fermion shadow of the clustering established from a gap in Part I and used throughout Part III.

5.2 Real structure and the tenfold way

The tenfold way arranges gapped free-fermion phases by their behavior under the antiunitary symmetries: time reversal T, particle–hole C, and their product, the chiral symmetry $S = TC$. Implemented on $\mathfrak{h}$, these are antilinear (for T, C) or linear (for S) operators squaring to $\pm\mathbf{1}$, and they endow the observable algebra with a *Real* structure in the sense of Atiyah: a real form $A^{\mathbb{R}} \subseteq \mathrm{CAR}(\mathfrak{h})$ (or a $\mathbb{Z}_2$-graded, Clifford-module refinement thereof) whose operator K-theory, in the degree fixed by the symmetry class, is the group of the Kitaev periodic table [10, 11, 12]. The eight real classes are carried by KO, the two complex classes by KU, and the eightfold Bott periodicity of KO is the eightfold periodicity of the table. The point for us is structural: the algebra that classifies free-fermion phases is a *real* C^*-algebra, and its invariant is an operator-K class, precisely the input that Aoki's theorem consumes.

Example 5.1 (The SSH chain as a K_1-class)**.** The Su–Schrieffer–Heeger chain [13] is the two-band model with single-particle Bloch Hamiltonian

$$H(k) = \big(t_1 + t_2 \cos k\big)\sigma_x + t_2 \sin k\, \sigma_y, \qquad q(k) = t_1 + t_2 e^{ik},$$

chiral-symmetric under $\sigma_z H(k) \sigma_z = -H(k)$ and gapped precisely when $|t_1| \neq |t_2|$. Chiral symmetry flattens the Fermi projection into the off-diagonal unitary $u(k) = q(k)/|q(k)|$, and the winding number

$$\nu = \frac{1}{2\pi i} \oint u(k)^{-1}\, du(k) \in \mathbb{Z}$$

is the homotopy class of $u : S^1 \to U(1)$. In operator-algebraic terms ν is the class $[u] \in K_1(A)$ of the observable algebra A — the translation-invariant algebra $C(S^1)$ in the clean case, or the real-space (possibly disordered) chiral algebra in general — and it is computed as the index of the associated Toeplitz operator, a genuine operator-K invariant [12, 17]. It takes $\nu = 1$ for $|t_1| < |t_2|$ and $\nu = 0$ for $|t_1| > |t_2|$. This is the flagship example of the program's SSH transition, and it will be the example on which the corollary of Section 6 is read.

The SSH class sits in K_1; the two-dimensional Chern insulator's Hall conductance sits in K_0 and is an integer multiple of e^2/h [14]; the disordered analogues sit in the K-theory of the crossed product of Section 7. In every case the invariant is a class in the operator K-theory of a real (or Real) observable algebra, and that is all Theorem II-B needs.

6 Solidification: operator K-theory as a condensed invariant

We come to the load-bearing result. Topological invariants of free-fermion and disordered phases are classes in the operator K-theory of a Banach or C^*-algebra of observables. Aoki's theorem shows that this operator K-theory is recovered from an intrinsically condensed operation applied to the *algebraic* K-theory of the condensed algebra. This section states that theorem with its hypotheses and draws the consequence for phases, taking care not to inflate it.

6.1 Aoki's theorem

We change convention for this section only: algebras are *real*. This is not a technicality to be waved away. It is exactly the setting in which the physically relevant KO-graded invariants live, so the real hypothesis is a feature. For a real Banach algebra A its condensation $\underline{A}$, $\underline{A}(S) = C(S, A)$, is a condensed (indeed solid) ring in the sense of Clausen–Scholze, and one may form its *connective* algebraic K-theory $K_{\mathrm{alg}}(\underline{A})$ as a condensed spectrum. The solidification functor Solid is the reflection onto solid condensed spectra of [1, 2].

Theorem II-B (Aoki's solidification theorem [5])**.** *Let A be a real associative algebra and $\underline{A}$ its condensation. The solidification of the* connective *algebraic K-theory of $\underline{A}$ is discrete and recovers the connective semitopological K-theory of A in the sense of Friedlander–Walker (and Blanc),*

$$\mathrm{Solid}\big(K_{\mathrm{alg}}(\underline{A})\big) \;\simeq\; K^{\mathrm{semi}}(A).$$

If moreover A is a real Banach algebra, this is the connective part *of the topological (operator) K-theory; the full, Bott-periodic operator K-theory is obtained only after inverting the Bott class β,*

$$K_{\mathrm{op}}(A) \;\simeq\; \mathrm{Solid}\big(K_{\mathrm{alg}}(\underline{A})\big)\big[\beta^{-1}\big],$$

an equivalence of spectra, where β is the generator of the relevant Bott group ($\beta \in K_2^{\mathrm{top}}(\mathbb{C}) \cong \mathbb{Z}$ in the complex case; the order-eight real Bott class in the KO case).

Two points of bookkeeping matter, and we make them explicit rather than bury them. First, the solidification is of the *connective* algebraic K-theory, so it lands in a connective spectrum with no homotopy in negative degrees, whereas operator K-theory is Bott-periodic and does have negative-degree groups. The two therefore agree only in nonnegative degrees; the equivalence with the full $K_{\mathrm{op}}(A)$ holds after the Bott inversion $[\beta^{-1}]$, and writing $K_{\mathrm{op}}(A) \simeq \mathrm{Solid}(K_{\mathrm{alg}}(\underline{A}))$ without it would be false in negative degrees. Second, the real versus Real (KR) distinction: β is the order-eight real Bott class, and the eightfold periodicity of KO is the eightfold structure of the real symmetry classes. A word on the topology in the first clause: A carries its given topology — discrete when none is specified, in which case $\underline{A}$ is the condensation of the underlying discrete algebra, whose value on a profinite S is the algebra of locally constant A-valued functions, a nontrivial condensed ring because profinite sets are richly disconnected, so the semitopological K-theory it computes is nontrivial even in the discrete case. The Banach clause, with its Bott inversion, is the one we use. The proof is Aoki's and we claim no part of it; what we contribute is the reading of the theorem inside the program.

6.2 Free-fermion invariants are condensed-mathematical

Corollary 6.1 (Provenance of the tenfold-way invariants)**.** *Let a free-fermion or disordered topological insulator or superconductor have real observable Banach algebra A — the real form of the (bulk or disorder-averaged) observable algebra in its symmetry class. Then its tenfold-way operator-K invariant $\nu \in K_{\mathrm{op}}(A)$ is the value at A of the composite functor*

$$A \;\longmapsto\; \underline{A} \;\longmapsto\; K_{\mathrm{alg}}(\underline{A}) \;\longmapsto\; \mathrm{Solid}\big(K_{\mathrm{alg}}(\underline{A})\big)\big[\beta^{-1}\big] \;\simeq\; K_{\mathrm{op}}(A) \ni \nu.$$

In particular the invariant factors through condensation, solidification, and the Bott inversion.

Concretely, for the SSH chain of Example 5.1 the winding number $\nu = [u] \in K_1(A)$ is a class in $\mathrm{Solid}(K_{\mathrm{alg}}(\underline{A}))[\beta^{-1}]$: the integer that distinguishes the two SSH phases is the value of the condensed-mathematical composite above. The number is the same integer computed in 1979; what Theorem II-B adds is that it is the output of solidification, followed by Bott inversion, applied to the algebraic K-theory of a condensed algebra.

Proof. By the K-theoretic classification of free-fermion and disordered phases, the symmetry-class invariant of such a system is a class in the operator K-theory of its (real/Real) observable algebra A, in the appropriate KO- or KU-degree [10, 11, 12]. Apply Theorem II-B to A: the operator K-theory group in which ν lives is $\mathrm{Solid}(K_{\mathrm{alg}}(\underline{A}))[\beta^{-1}]$ (the Bott-inverted, periodic value), so ν is a class in the target of the displayed composite, and the composite computes it. □

Remark 6.2 (Exactly how much this claims)*.* It claims that the invariant's *home* and *construction route* are condensed-mathematical, not that the invariant is new or numerically different. A Chern number computed as a solid K-class equals the Chern number computed by integrating a curvature form. The content is that the two mature theories (the operator K-theory of topological insulators and the condensed/solid K-theory of Clausen–Scholze and Aoki) meet here, so the family-theoretic, descent-theoretic, and profinite-disorder machinery of the condensed side becomes available for an invariant that was previously computed by hand. This is the sense in which the program's value is organizational. We state it this way in deference to the honest appraisal that the program does not, on its own, produce a different Chern number.

Remark 6.3 (Why the real setting is the right one)*.* The Kitaev periodic table is KO-graded: the eightfold Bott periodicity of real K-theory is the eightfold structure of the real symmetry classes, with the two complex classes carried by KU. Aoki's theorem is proved for real algebras, so it lands precisely where the physics lives; a complex-only statement would miss the time-reversal and particle–hole classes. The refinement to a genuinely Real (KR) statement matching all ten classes is Conjecture II-3.

7 Disorder, crossed products, and profinite probes

Disorder is where the condensed language stops repackaging manifolds and starts matching the physical configuration space. A disordered family is literally a point of the Hamiltonian moduli object over a profinite base.

7.1 The disorder hull as a profinite probe

Let Q be a finite set of local configurations[1] and

$$\Omega \;=\; Q^{\mathbb{Z}^d},$$

the space of configurations, with the product topology. It is compact, Hausdorff, and totally disconnected (a profinite set), and the shift action of $\mathbb{Z}^d$ by translations makes it a profinite $\mathbb{Z}^d$-space. Restricting a configuration to a finite window $\Lambda_i \Subset \mathbb{Z}^d$ gives finite quotients $\Omega \twoheadrightarrow \Omega_i = Q^{\Lambda_i}$, and $\Omega = \varprojlim_i \Omega_i$.

[1]The prospectus and the program's notation write this alphabet as F; we use Q to avoid collision with the Nachtergaele–Sims–Young F-function of Part I (carried here only inside the interaction Banach space $\mathcal{B}_F$).

Proposition 7.1 ($C(\Omega)$ is a commutative AF algebra). *For $\Omega = \varprojlim_i \Omega_i$ profinite, the pullbacks along $\Omega \to \Omega_i$ induce an isometric $*$-isomorphism*

$$C(\Omega) \;\cong\; \varinjlim_i C(\Omega_i),$$

so $C(\Omega)$ is a separable, commutative, unital AF algebra. Its state space is the weak-$*$ compact convex set $\mathrm{Prob}(\Omega)$ of Borel probability measures on Ω, and its condensation satisfies $\underline{C(\Omega)}(S) = C(S \times \Omega)$ for profinite S; in particular Ω itself is an admissible probe.

Proof. Each $\Omega_i = Q^{\Lambda_i}$ is finite, so $C(\Omega_i) \cong \mathbb{C}^{|\Omega_i|}$ is finite-dimensional. The pullback maps $C(\Omega_i) \to C(\Omega_j)$ for $i \le j$ are unital $*$-monomorphisms, and any $g \in C(\Omega)$ is a uniform limit of functions factoring through some Ω_i: the cylinder functions are dense by Stone–Weierstrass, since they separate points of Ω and are closed under the algebra operations and conjugation. Hence the natural map $\varinjlim_i C(\Omega_i) \to C(\Omega)$ is an isometric $*$-isomorphism onto a dense, hence closed and therefore full, subalgebra. This exhibits $C(\Omega)$ as an AF algebra; commutativity and unitality are clear, and separability follows as in Proposition 2.2. By Riesz representation, states on $C(\Omega)$ are Borel probability measures. For the condensation, $\underline{C(\Omega)}(S) = C(S, C(\Omega)) = C(S \times \Omega)$ by the exponential law for continuous maps out of the compact space S into $C(\Omega)$ (equivalently, the tensor identity $C(S) \otimes C(\Omega) \cong C(S \times \Omega)$ for compact Hausdorff spaces). □

7.2 The Bellissard crossed product and its condensation

The observables of a homogeneous disordered system are not functions on Ω alone but the covariant operators; the algebra that carries the physics is the crossed product.

Theorem II-C (Crossed-product disorder algebra). *Let $\Omega = Q^{\mathbb{Z}^d}$ with the shift action T of $\mathbb{Z}^d$. The covariant observable algebra of a homogeneous disordered family is the crossed product*

$$\mathfrak{A} \;=\; C(\Omega) \rtimes_T \mathbb{Z}^d,$$

a separable unital C^*-algebra. Its K-theory $K_*(\mathfrak{A})$ carries the disorder-averaged topological invariants of the family, and an invariant ergodic measure induces a trace whose range on K_0 is the gap-labelling group. The condensation $\underline{\mathfrak{A}}$ is functorial in the finite quotients of Ω: a T-equivariant quotient $\Omega \to \Omega'$ induces a $*$-homomorphism $\mathfrak{A}' \to \mathfrak{A}$ and hence a morphism $\underline{\mathfrak{A}'} \to \underline{\mathfrak{A}}$ of condensed C^*-algebras, compatibly with composition.

Proof. $C(\Omega)$ is a separable unital C^*-algebra (Proposition 7.1) and $\mathbb{Z}^d$ is countable discrete and *amenable*, so the full and reduced crossed products coincide and $\mathfrak{A} = C(\Omega) \rtimes_T \mathbb{Z}^d$ is an unambiguous separable unital C^*-algebra by the standard construction; we write $\rtimes$ for this common value. That its K-theory computes the disorder-averaged invariants, and that a T-invariant ergodic probability measure $\mathbb{P}$ on Ω induces a faithful trace $\tau_{\mathbb{P}}$ on $\mathfrak{A}$ whose composition with the K_0-pairing is the integrated density of states and yields the gap-labelling group, is Bellissard's noncommutative geometry of the quantum Hall effect [15, 16, 12]. For functoriality, a T-equivariant continuous surjection $p : \Omega \to \Omega'$ induces a unital equivariant $*$-monomorphism $p^* : C(\Omega') \hookrightarrow C(\Omega)$; by the universal property of the crossed product an equivariant $*$-homomorphism of coefficient algebras induces a $*$-homomorphism $p^* \rtimes \mathbb{Z}^d : \mathfrak{A}' = C(\Omega') \rtimes_T \mathbb{Z}^d \to C(\Omega) \rtimes_T \mathbb{Z}^d = \mathfrak{A}$, functorial in p. Applying condensation, which is a functor (Proposition 2.3), gives $\underline{\mathfrak{A}'} \to \underline{\mathfrak{A}}$, compatible with composition. Since $\Omega = \varprojlim_i \Omega_i$, these assemble the condensation of $\mathfrak{A}$ over the inverse system of finite quotients. $\square$

Remark 7.2 (Bulk–boundary and coarse geometry). The K-theory of $\mathfrak{A}$ is not only a receptacle for a number; the bulk–boundary correspondence realizes the boundary invariant as the image of the bulk class under the connecting map of the Toeplitz extension of $\mathfrak{A}$ [17], and the coarse/Roe-algebraic picture gives the same invariants a controlled-topology description [18]. Each of these is a K-theory computation to which Corollary 6.1 applies, so the boundary invariant is condensed for the same reason the bulk one is. The naturality of the *solid* invariant in the disorder hull (that solidification commutes with the maps above) is the content of Conjecture II-2.

8 Conjectures

The theorems above are the provable core. The program's ambition on the positivity/C^* side is larger, and we state the open part as numbered conjectures with stable identifiers. Each is separated cleanly from what is proved, and each names the nearest rigorous result it would extend.

Conjecture II-1 (Positivity cuts out a closed condensed substack)**.** *The assignment $H \mapsto (\mathcal{A}, \mathcal{S}(\mathcal{A}))$ upgrades to a morphism of condensed stacks from $\mathfrak{Ham}_{d,G}$ to the stack of condensed unital C^*-algebras equipped with their condensed state space. Moreover, inside a larger "formal-interaction" stack in which the C^*-relations are dropped, positivity together with the C^*-identity $\|a^*a\| = \|a\|^2$ cuts out $\mathfrak{Ham}_{d,G}$ as a* closed *condensed substack.*

This would promote Theorem II-A from a statement about a single algebra to one about the whole moduli object; the closedness is the stacky form of "positivity is a closed condition," and its natural proof would run through the condensed-stack technology of [3] together with the condensed-set morphism $\mathbb{R} \times \underline{\mathcal{B}_F} \to \underline{\mathrm{Aut}(\mathcal{A})}$ of Part I [23].

Conjecture II-2 (Naturality of the solid invariant in disorder)**.** *For a disordered family $H \in \mathfrak{Ham}_{d,G}(\Omega)$ with observable algebra $\mathfrak{A} = C(\Omega) \rtimes_T \mathbb{Z}^d$, the solid invariant* $\mathrm{Solid}(K_{\mathrm{alg}}(\underline{\mathfrak{A}}))$ *agrees with the operator-K invariant* $K_{\mathrm{op}}(\mathfrak{A})$ naturally in Ω*: the identifications of Theorem II-B commute with the $*$-homomorphisms induced by T-equivariant finite quotients of Theorem II-C, so the invariant is compatible with every finite-resolution approximation of the hull.*

This is the disorder-averaged, family version of Corollary 6.1. It is the precise sense in which "profinite probes match the configuration space": the invariant is not merely definable over Ω but is the limit of its values on the finite quotients Ω_i.

Conjecture II-3 (Real/KKO refinement of the periodic table)**.** *The KO-graded Kitaev periodic table is recovered by solidification of the* Real *(in the sense of Atiyah's KR) algebraic K-theory of the condensed real observable algebra: for each of the ten symmetry classes, the class-appropriate KR-degree of* $\mathrm{Solid}(K_{\mathrm{alg}}(\underline{A}))$ *equals the operator KKO-invariant of the system, and the eightfold real periodicity matches the eightfold structure of the real classes.*

Theorem II-B is a real, not yet a Real, statement, and the tenfold way needs the KR grading that keeps track of the anti-linear time-reversal and particle–hole symmetries [12, 17, 11]. The conjecture is that Aoki's solidification is compatible with the Real structure so that the whole table, not just its complex and real-linear parts, is condensed.

Conjecture II-4 (Solid state space as a classifying object)**.** *The condensed state space $\underline{\mathcal{S}(\mathcal{A})}$, with the convex structure making it a solid $\mathbb{R}$-convex object, is a classifying object for condensed cyclic representations: a continuous family of states over a profinite base S is the same as a map $S \to \underline{\mathcal{S}(\mathcal{A})}$, and the GNS functor of Proposition 4.2 descends to a universal condensed field of representations over $\underline{\mathcal{S}(\mathcal{A})}$, refining the continuity of Theorem 4.3 to a continuous field of Hilbert spaces on the locus where the GNS dimension is locally constant.*

This is the condensed form of the state-space approach to phase classification, in which phases are homotopy classes of sections of bundles of (pure) states [21, 22]; the conjecture is that those bundles and their sections are the shadow of a single condensed classifying object, and that GNS is its universal representation.

Conjecture II-5 (Condensed enhancement of the split property and superselection)**.** *The split property of gapped ground states and the Doplicher–Haag–Roberts superselection structure admit condensed enhancements: the sector structure — a symmetric or braided tensor category in the relevant dimension — is detected by a condensed invariant over profinite bases, and the split property provides the condensed factorization along which π_0 of the phase stack descends through finite quotients of a disorder hull.*

The split property is central to the operator-algebraic classification of gapped phases and to the construction of the indices that make that classification work [19, 20]. The conjecture asks that this structure be visible to the profinite probes, so that the descent of phase labels along disorder approximations (the theme of Theorem II-C and Conjecture II-2) has a representation-theoretic counterpart.

9 Discussion and limitations

Scope of the solid bridge. Theorem II-B and Corollary 6.1 reach the invariants of free-fermion and disordered phases (the invertible, K-theoretic sector). They say nothing about noninvertible topological order, whose excitations form a braided or modular tensor category rather than a K-theory class. A condensed account of that order would need a condensed *higher* stack of phases and defects, not a K-theory spectrum, and it is out of scope here; Parts V and VI record the boundary of what the K-theoretic story can classify. Within its scope, the bridge is exact, but its scope is the invertible sector.

Real versus complex, connective versus periodic. We have been deliberate about Aoki's hypotheses. The theorem is about real algebras; the operator K-theory it recovers is KO-flavored, which is the right flavor for the periodic table, but the identifications carry connective-versus-periodic and real-versus-Real bookkeeping that we have imported rather than restated. A reader who wants the graded-group form must read it off the source with those conventions in place. Overstating the theorem as a naked Bott-periodic isomorphism would be a mistake, and we have avoided it.

Unital versus nonunital. Our systems have finite on-site dimension, so $\mathcal{A}$ is unital and $\mathcal{S}(\mathcal{A})$ is compact: the hypotheses of Lemma 3.1 and Theorem II-A hold on the nose. For systems whose observable algebra is nonunital (for instance certain half-space or continuum models) the state space is only weak-$*$ locally compact, and one must pass to the multiplier algebra or unitization before condensing; Theorem II-A then applies to the unitization. We flag this rather than hide it.

Light versus full condensed. Separability of $\mathcal{A}$ and of $C(\Omega)$ lets us work with light condensed sets throughout, which sidesteps the set-theoretic size issues of the unrestricted theory and is exactly matched to the countable inverse limits that describe finite-resolution disorder data. For systems that are genuinely non-separable this convenience is lost and the full condensed formalism, with its universe bookkeeping, is required.

What is genuinely new here. Nothing in Sections 2 to 4 and 7 is a new theorem of operator algebras; the individual facts are classical or cited. The new element is their assembly: the state space, positivity, GNS, and the crossed product are placed in the condensed category so that Aoki's bridge can act on them, and the ground-state section is shown to be a genuine condensed point over the gapped locus. The synthesis is the contribution, and its honest measure is that it makes an existing invariant condensed, not that it makes a new one.

10 Conclusion

We have carried the observable side of a quantum lattice system (its quasi-local C^*-algebra, its positivity structure, its state space, and its K-theory) into condensed mathematics, and we have been explicit about the line between theorem and conjecture. On the theorem side: the quasi-local algebra is a separable AF C^*-algebra whose condensation is a light sheaf of C^*-algebras (Propositions 2.2 and 2.3); the state space condenses to a compact Hausdorff condensed set on which positivity is a closed condition (Theorem II-A); GNS is a functor and the ground-state section over a uniformly gapped family is weak-$*$ continuous (Proposition 4.2 and theorem 4.3); Aoki's solidification theorem makes the operator-K invariant of a real Banach observable algebra a condensed-mathematical object (Theorem II-B and corollary 6.1); and the disorder crossed product condenses functorially in the finite quotients of the profinite hull (Proposition 7.1 and Theorem II-C). On the conjecture side we have named five open problems (Conjectures II-1 to II-5) whose resolution would complete the positivity/C^* module of the program.

The single load-bearing insight is worth isolating. A topological invariant of a free-fermion or disordered phase is an operator-K class, and by Aoki's theorem that class is the Bott-inverted solidification of the connective algebraic K-theory of a condensed algebra. The invariant is unchanged; its provenance is now condensed, and with that provenance comes the family-theoretic and profinite-disorder machinery that the rest of the program deploys. That is the whole of what this paper claims, and it is stated in the key of a research program rather than a finished theory.

Code availability

The formal-verification code accompanying this paper (finite-dimensional C^*-algebra positivity checks, the explicit GNS construction, and the Bloch-ball geometry of qubit states) is at github.com/YonedaAI/topological-phases-of-matter, in `src/positivity-cstar-norms/`.

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