Topological Phases of Matter in the Condensed-Mathematics Paradigm: A Modular Research Program

Part VI of *Topological Phases of Matter in the Condensed-Mathematics Paradigm*

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Abstract

This synthesis assembles the five analytic and topological modules of the preceding Parts into one research program: topological phases of matter, read as the components of a condensed moduli stack of uniformly gapped local Hamiltonians. The spine is a single sequence: local quantum interactions $\rightsquigarrow$ condensed moduli stack $\mathfrak{Ham}_{d,G}$ $\rightsquigarrow$ uniformly gapped substack $\mathfrak{Gap}_{d,G}$ $\rightsquigarrow$ stabilized phase ∞-groupoid $\mathfrak{Phase}_{d,G}$ $\rightsquigarrow$ invertible condensed phase spectrum $\mathbf{IP}^{\mathrm{cond}}_{d,G}$, and each arrow is owned by a Part. Part I makes the interaction space a condensed Banach object and the Heisenberg flow a condensed morphism with a base-uniform Lieb–Robinson light cone; Part II attaches the quasi-local C^*-algebra, its compact condensed state space, and the solid K-theory invariant obtained from Aoki's solidification theorem (correctly, only after inverting the Bott class); Part III cuts out the uniformly gapped substack and proves its stability on the frustration-free / LTQO stratum against the wall of Cubitt–Pérez-García–Wolf undecidability; Part IV group-completes the invertible sector into the condensed spectrum and sets it against the Freed–Hopkins bordism target; Part V asks which spectral classes a lattice model realizes, bounded by the commuting-projector no-gos of Kapustin–Fidkowski and Kapustin–Spodyneiko. We treat the program as a *modular composition*: each Part is a module whose output is the next module's input, and structure emerges at each level: functorial invariants of disordered families, a robust set of phases, the higher homotopy of pumps and defects, an obstruction subgroup of unrealized classes. We develop the transition calculus, in which a topological phase transition is the crossing of the gapless discriminant Σ_f and its charge is the relative class $\partial\nu \in E^{q+1}(B, B \setminus \Sigma_f)$, an obstruction detected by linking spheres; we work the SSH chain through all five modules; and we run the profinite disorder thread $\Omega = F^{\mathbb{Z}^d}$

through the whole. The module-level results are theorems, cited to Parts I–V. The program-level statements (that the modules glue into a single condensed higher stack with $\mathbf{IP}^{\mathrm{cond}}_{d,G}$ its invertible spectrum, that the microscopic and field-theoretic classifications agree, that the relative charge is a spectral boundary map, that all invariants descend along disorder hulls) are stated as numbered Conjectures VI-1 through VI-4 and never as proven. A consolidated table records all nineteen module conjectures with their dependencies. The genuinely new thesis is not a different Chern number; it is a single categorical and homological environment for continuous families, profinite disorder, completions, symmetry, stacking, defects, and transition loci.

Contents

1 Introduction

1.1 The proposal

A quantum phase is usually presented as an entry in a classification table. This program takes a different starting point, due in outline to the seed prospectus of the series and realized in detail across Parts I–V: a phase is a *component of a condensed moduli object of uniformly gapped local Hamiltonians*. Fix a spatial dimension d, a lattice or coarse metric space L, on-site Hilbert spaces, a locality (interaction-decay) class, and an internal or spatial symmetry group G. Let $\mathcal{I}_{d,G}$ be the space of admissible G-symmetric local or quasi-local interactions. Rather than use $\mathcal{I}_{d,G}$ as an ordinary topological space, pass to its condensation

$$X \longmapsto \underline{X}, \qquad \underline{X}(S) = \mathrm{Cont}(S, X), \qquad S \text{ profinite}, \tag{1}$$

a sheaf on the site of profinite sets with finite jointly surjective covers. On compactly generated spaces the passage $X \mapsto \underline{X}$ is fully faithful [36, 5], so coupling tori, Banach interaction spaces, and compact disorder hulls are not discarded but embedded into a category with exact homological algebra. Because Hamiltonians carry gauge and quasi-local automorphisms, the right object is a condensed *anima* or *stack*, not a condensed set.

The whole program is the study of one geometric object and one substack. Over a profinite probe S let $\mathfrak{Ham}_{d,G}(S)$ be the S-continuous families of G-symmetric quasi-local interactions, and let

$$\mathfrak{Gap}_{d,G}(S) = \bigcup_{\Delta>0} \{\, H \in \mathfrak{Ham}_{d,G}(S) : \inf_{s\in S} \mathrm{gap}(H_s) \geq \Delta \,\} \tag{2}$$

be the *uniformly* gapped substack (the word uniformly being load-bearing, since a family whose pointwise gaps degenerate to zero must be excluded). Inverting a class $\mathcal{W}$ of gapped adiabatic / quasi-local equivalences and stabilizing by trivial ancillas produces the phase ∞-groupoid, and its shape records the phases:

$$\mathfrak{Phase}_{d,G} := \left(\mathfrak{Gap}_{d,G}[\mathcal{W}^{-1}]\right)^{\mathrm{st}}, \qquad \mathrm{Phases}_{d,G} = \pi_0\, \mathrm{Shape}(\mathfrak{Phase}_{d,G}). \tag{3}$$

Stacking systems, $\boxtimes$, makes $\mathfrak{Phase}_{d,G}$ symmetric monoidal; group-completing its invertible sector yields the connective condensed spectrum $\mathbf{IP}^{\mathrm{cond}}_{d,G}$. The two slogans of the program are the anchors of everything below:

$$\boxed{\text{topological phase} = \text{a component of the stabilized condensed stack of gapped systems}} \tag{4}$$

and

$$\boxed{\begin{array}{c}\text{topological transition} = \\ \text{crossing the gapless discriminant in the Hamiltonian moduli stack.}\end{array}} \tag{5}$$

1.2 The program as a modular hierarchy

What Parts I–V supply is not a monolith but five interoperable *modules*, each of which takes the previous module's output as its input and produces new structure. This is the sense in which the program is modular: the composition is a hierarchy, and at each level of the hierarchy a property emerges that was not visible one level down.

1. **Locality (Part I [25]).** Makes $\mathcal{I}_{d,G}$ a real Banach space $\mathcal{B}_F$ through a Nachtergaele–Sims–Young F-function locality norm, and promotes the Heisenberg dynamics to a morphism of condensed sets with a *base-uniform* Lieb–Robinson light cone. Emergent structure: the ground floor $\mathfrak{Ham}_{d,G}$ exists as a condensed object with a well-defined notion of quasi-local time evolution over any profinite base.

2. **Positivity and C^*-norms (Part II [28]).** Attaches the quasi-local C^*-algebra $\mathcal{A}$, its compact condensed state space, and, via Aoki's solidification theorem, a *solid* K-theory invariant. Emergent structure: functorial topological invariants of disordered families over profinite hulls, produced by an intrinsically condensed operation.

3. **The gap (Part III [29]).** Cuts out the uniformly gapped substack $\mathfrak{Gap}_{d,G}$ and proves its stability where a stability theory holds with uniform constants. Emergent structure: a robust set of phases $\mathrm{Phases}_{d,G}$, once $\mathcal{W}$ is inverted and ancillas are added.

4. **Stabilization and effective field theory (Part IV [26]).** Group-completes the invertible sector into the connective condensed spectrum $\mathbf{IP}^{\mathrm{cond}}_{d,G}$ and compares it with the Freed–Hopkins bordism classification. Emergent structure: the higher homotopy of pumps and defects, and the relative charge $\partial\nu$ as a spectral boundary map.

5. **Realizability (Part V [27]).** Identifies which spectral classes are produced by an explicit gapped lattice Hamiltonian. Emergent structure: an obstruction subgroup separating abstract classes from lattice-realizable ones.

Two threads cut across all five modules and are what make the composition cohere rather than merely stack. The first is the *transition calculus* $(U_f, \Sigma_f, \nu_f, \partial\nu)$, which ties modules III–V together: the gapped locus of a family, its gapless discriminant, the locally constant phase label, and the relative charge of a crossing. The second is the *profinite disorder thread* $\Omega = F^{\mathbb{Z}^d}$, which runs through every module: descent of Lieb–Robinson estimates in Part I, crossed-product observable algebras in Part II, uniform-gap descent in Part III, parametrized invariants in Part IV, and functorial realizability in Part V.

1.3 What is proved and what is conjectured

The stance of this synthesis is the stance of the whole series: a rigorous research *program*, not a finished theory. We are strict about the boundary. The *module-level* results (Theorems I-A through V-C, and the citations of Aoki, Kubota, Freed–Hopkins, Ogata, and Kapustin–Fidkowski as rigorous anchors) are theorems, and we restate them (as Recollections, with citations to the Parts that prove them) rather than reprove them. The *program-level* claims are open. That the modules glue into a single condensed higher stack with $\mathbf{IP}^{\mathrm{cond}}_{d,G}$ as its invertible spectrum; that the microscopic and field-theoretic classifications agree; that the relative charge is the boundary map of that spectrum; that every program invariant descends along disorder hulls; each of these is stated below as a numbered Conjecture VI-k and is never asserted as proven.

Two hard theorems bound the program from outside and are quoted wherever they bite. Cubitt–Pérez-García–Wolf [10] proved that the spectral gap is undecidable, so there is no algorithm and no uniform criterion deciding membership in $\mathfrak{Gap}_{d,G}$; existence of a thermodynamic gap is a *hypothesis that defines* the substack, not a computable property. Two commuting-projector no-gos bound realizability: Kapustin–Fidkowski [18] proved that a $U(1)$-symmetric local commuting-projector Hamiltonian has zero electric Hall conductance ($\sigma_H = 0$), and Kapustin–Spodyneiko [21] proved that its chiral central charge vanishes ($c_- = 0$) with no symmetry hypothesis; so every chiral invertible class (nonzero σ_H under $U(1)$, or nonzero c_-) lies outside the commuting-projector image. Neither wall is ever crossed in what follows.

1.4 Relation to companion papers

This paper is the capstone of a six-part series and consumes the outputs of Parts I–V; it imports the canonical notation of the series verbatim and does not restate it. We summarize each companion in one paragraph, and Section 3 develops the composition in full.

Part I, Condensed Locality [25], is the analytic ground floor. Fixing an F-function it makes the G-symmetric interactions of finite F-norm a real Banach space $\mathcal{B}_F$, condenses it to $\underline{\mathcal{B}_F}(S) = \mathrm{Cont}(S, \mathcal{B}_F)$, and proves that Heisenberg dynamics is a strongly continuous group of $*$-automorphisms obeying a Lieb–Robinson bound (Theorem I-A), that its assignment is a morphism of light condensed sets (Theorem I-B), and that a profinite family is exactly compatible finite data with a base-uniform light cone (Theorem I-C). It generates the equivalence class $\mathcal{W}$ from quasi-adiabatic continuation and states Conjectures I-1 (condensed higher stack), I-2 (condensed group of quasi-local automorphisms), and I-3 (descent of Lieb–Robinson estimates along a disorder hull).

Part II, Positivity, C^*-Norms, and Condensed State Spaces [28], is the observable side. It shows the state space $\mathcal{S}(\mathcal{A})$ condenses to a compact Hausdorff condensed set cut out by positivity and normalization (Theorem II-A), that Aoki's solidification recovers operator K-theory after Bott inversion (Theorem II-B), and that Bellissard's crossed product $C(\Omega) \rtimes \mathbb{Z}^d$ condenses functorially in the finite quotients of the hull (Theorem II-C). It records Conjectures II-1 through II-5 (closed condensed substack; naturality of the solid invariant; Real/KKO refinement; solid classifying object; condensed superselection). Following Part II we write the disorder alphabet as Q, not F, to avoid collision with the F-function of Part I;

see Section 8.

Part III, The Uniformly Gapped Substack [29], is the gap module. It proves that the finite-volume gap is Lipschitz and the gapped locus is open at finite resolution (Theorem III-A), that $\mathfrak{Gap}_{d,G}$ is condensed-open on the frustration-free / LTQO stratum (Theorem III-B), and that a uniform gap forces uniform clustering (Theorem III-C), all against the CPW undecidability wall; it states Conjectures III-1 (openness on the physical stratum), III-2 (uniform-gap descent), and III-3 (quasi-adiabatic continuation realizes $\mathcal{W}$).

Part IV, From Lattice Models to Effective Field Theories [26], is the stabilization module. It makes $\boxtimes$ a symmetric-monoidal structure, group-completes $\mathrm{Phases}_{d,G}$, assembles $\mathbf{IP}^{\mathrm{cond}}_{d,G}$ with Kubota's Ω-spectrum as its rigorous carrier, and sets the lattice classification against Freed–Hopkins. It states Conjectures IV-1 (condensed/solid refinement), IV-2 (lattice–EFT comparison equivalence), IV-3 (relative charge as spectral boundary map), IV-4 (solidification commutes with group completion), and IV-5 (renormalization functor).

Part V, Physical Realizability [27], is the realizability module. It proves that every group-cohomology class is realized by a commuting-projector model (Theorem V-A), completeness of the operator-algebraic index in $d = 1$ for on-site finite symmetry, with the $d = 2$ index surjective but its completeness open (Theorem V-B and Remark V-B′), and the two commuting-projector no-gos, electric (Kapustin–Fidkowski) and thermal/chiral (Kapustin–Spodyneiko) (Theorem V-C). It states Conjectures V-1 (realizability image is the SRE subspectrum), V-2 (chiral realizability by non-commuting models), and V-3 (profinite-family realizability).

1.5 Outline

Section 2 recalls the condensed-mathematics substrate and fixes conventions. Section 3 is the spine: the boxed sequence, arrow by arrow, with each module's theorems restated and cited and its emergent structure named. Section 4 makes modular composition explicit: the composition diagram, the emergent properties, and the two walls. Section 5 states the homotopy dictionary. Section 6 develops the relative-charge formalism. Section 7 works the SSH chain through all five modules. Section 8 runs the profinite disorder thread. Section 9 states the program-level Conjectures VI-1 through VI-4 and consolidates all nineteen module conjectures with dependencies. Section 10 weighs what the paradigm contributes and what it does not, and Section 11 concludes.

2 The condensed-mathematics substrate

We recall only what the composition uses; Parts I and II give the careful development.

2.1 Condensation and profinite probes

A condensed set is a sheaf on the site of profinite sets S with covers the finite jointly surjective families [36]; the pyknotic variant of Barwick–Haine [5] differs only in set-theoretic bookkeeping. The functor (1) embeds compactly generated topological spaces fully faithfully, so no information is lost in passing from X to $\underline{X}$. The point of the passage is algebraic: condensed abelian groups form a Grothendieck abelian category with exact products and derived functors, where ordinary topological abelian groups do not, and this is precisely the environment in which derived invariants, descent, and completions behave. A condensed object is *light* when it is generated under the site by a countable family (the case for separable Banach spaces and separable C^*-algebras), which is why Parts I and II take care to work with separable data.

2.2 Stacks, anima, and shape

Because a Hamiltonian carries automorphisms (gauge transformations, on-site symmetry actions, quasi-local unitaries), the moduli problem $S \mapsto \{S\text{-families of Hamiltonians}\}$ valued in groupoids (or ∞-groupoids) is a candidate *condensed stack*, not merely a condensed set. Whether it satisfies descent along profinite covers is the content of Conjecture I-1 and is not proved. The *shape* Shape$(-)$ of a condensed anima is its underlying homotopy type; $\pi_0\,\mathrm{Shape}$ extracts the ordinary set of components, hence $\mathrm{Phases}_{d,G}$ in (3). Throughout, fraktur denotes the moduli stacks ($\mathfrak{Ham}, \mathfrak{Gap}, \mathfrak{Phase}$), calligraphic the interaction space and the equivalence class ($\mathcal{I}, \mathcal{W}$), boldface the spectra ($\mathbf{IP}^{\mathrm{cond}}$), and $\underline{(-)}$ condensation.

2.3 The boxed sequence

The program is summarized by one sequence, quoted from the prospectus and realized in Parts I–V:

$$\boxed{\begin{array}{c} \text{local quantum interactions} \\ \big\downarrow \text{ Part I} \\ \text{condensed moduli stack of Hamiltonians } \mathfrak{Ham}_{d,G} \\ \big\downarrow \text{ Part II} \\ \text{condensed observable algebras and solid } K\text{-theory} \\ \big\downarrow \text{ Part III} \\ \text{uniformly gapped substack } \mathfrak{Gap}_{d,G} \\ \big\downarrow \text{ Part IV} \\ \text{stabilized phase } \infty\text{-groupoid } \mathfrak{Phase}_{d,G} \\ \big\downarrow \text{ Part IV} \\ \text{invertible condensed phase spectrum } \mathbf{IP}^{\mathrm{cond}}_{d,G} \\ \big\downarrow \text{ Part V} \\ \text{realized subspectrum inside } \mathbf{IP}^{\mathrm{cond}}_{d,G}. \end{array}} \tag{6}$$

The prospectus states the middle five lines; Parts II and V add the observable-algebra and realization refinements. Each arrow is a module. Section 3 treats them in turn.

3 The spine: the boxed sequence and its five modules

We walk down (6). For each module we recall its principal theorems (as Recollections, attributed to the Part that proves them) and name the structure that emerges. Nothing in this section is claimed as new; the novelty of the synthesis is the composition, treated in Section 4.

3.1 Arrow I: local interactions to the condensed stack

The first arrow turns a physicist's list of local couplings into a condensed object that can be probed by profinite sets. Part I fixes an F-function F encoding interaction decay and defines the interaction Banach space $\mathcal{B}_F$.

Recollection 3.1 (Banach interaction space and condensed dynamics; Part I, Theorems I-A and I-B [25])**.** *$\mathcal{B}_F$ is a real Banach space; for $\Phi \in \mathcal{B}_F$ the infinite-volume Heisenberg dynamics $\tau_t^{\Phi} = \lim_{\Lambda\uparrow L} \tau_t^{\Phi,\Lambda}$ exists in operator norm, uniformly for t in compact sets, and is a strongly continuous one-parameter group of $*$-automorphisms of the quasi-local algebra $\mathcal{A}$ obeying a Lieb–Robinson bound. Its condensation*

$$\underline{D} : \underline{\mathbb{R}} \times \underline{\mathcal{B}_F} \longrightarrow \underline{\mathrm{Aut}(\mathcal{A})}, \qquad (t,\Phi) \mapsto \tau_t^{\Phi},$$

is a morphism of (light, when the interactions are separable) condensed sets.

Recollection 3.2 (Base-uniform light cone; Part I, Theorem I-C [25])**.** *Let $S = \varprojlim_i S_i$ be profinite and $\Phi_\bullet \in \mathrm{Cont}(S, \mathcal{B}_F) = \underline{\mathcal{B}_F}(S)$. Then $\Phi_\bullet$ is a norm limit of families factoring through finite quotients $S \to S_i$; $B = \sup_{s\in S}\|\Phi_s\|_F < \infty$; and if $\sup_s\|\Phi_s\|_{F_a} < \infty$ for a reweighting $F_a(r) = e^{-ar}F(r)$, the Lieb–Robinson bound holds for all $s \in S$ simultaneously with one velocity $v = 2\sup_s\|\Phi_s\|_{F_a} C_{F_a}/a$.*

Emergent structure. The ground floor $\mathfrak{Ham}_{d,G}$ is a condensed object over which time evolution is quasi-local *uniformly in the probe.* Uniformity is what makes every later module's "family over a base" well-posed: it is the reason a disordered family has a single light cone, a single clustering length, a single set of constants. Part I's Conjectures I-1, I-2, I-3 mark the edge: stack descent, a condensed automorphism group, and Lieb–Robinson descent along a hull; none proved.

3.2 Arrow II: observable algebras and solid K-theory

The second arrow attaches to the stack the algebra of observables and the K-theory from which topological invariants are built. For a spin system with finite on-site dimension $\mathcal{A}$ is a separable unital AF algebra (UHF in the homogeneous case), and its condensation is a sheaf of C^*-algebras, light because $\mathcal{A}$ is separable.

Recollection 3.3 (Condensed state space; Part II, Theorem II-A [28])**.** *For a unital C^*-algebra A, the state space $\underline{\mathcal{S}(A)}$ is a compact Hausdorff condensed set on which condensation is fully faithful, and positivity together with normalization exhibit it as a closed condensed subobject of the condensed dual ball. On a uniformly gapped family the ground-state section is weak-$*$ continuous by the Bachmann–Michalakis–Nachtergaele–Sims spectral-flow cocycle [4], a genuine point of $\underline{\mathcal{S}(A)}$ over the gapped locus.*

The load-bearing bridge is Aoki's theorem, and the honesty of the whole program depends on stating it with its Bott-inversion caveat, exactly as Part II now does.

Recollection 3.4 (Solidification and operator K-theory; Part II, Theorem II-B, after Aoki [3, 28])**.** *Let A be a real associative algebra and $\underline{A}$ its condensation. Solidification of the* connective *algebraic K-theory of $\underline{A}$ is discrete and recovers the connective semitopological K-theory of A (Friedlander–Walker, and Blanc),*

$$\mathrm{Solid}\big(K_{\mathrm{alg}}(\underline{A})\big) \;\simeq\; K^{\mathrm{semi}}(A).$$

If A is a real Banach algebra this is the connective part of operator K-theory; the full, Bott-periodic operator K-theory is recovered only after inverting the Bott class β,

$$K_{\mathrm{op}}(A) \;\simeq\; \mathrm{Solid}\big(K_{\mathrm{alg}}(\underline{A})\big)\big[\beta^{-1}\big], \tag{7}$$

with $\beta \in K_2^{\mathrm{top}}(\mathbb{C}) \cong \mathbb{Z}$ complex, the period-eight real Bott class in the KO case (an infinite-order generator of the eightfold KO periodicity).

We stress the bookkeeping because it is easy to get wrong and consequential when composed. Writing $K_{\mathrm{op}}(A) \simeq \mathrm{Solid}(K_{\mathrm{alg}}(\underline{A}))$ without the inversion $[\beta^{-1}]$ is false in negative degrees: solidification of connective algebraic K-theory lands in a connective spectrum with no negative homotopy, whereas operator K-theory is periodic. The two agree in nonnegative degrees, which is all the SSH winding (a degree-≥ 0 class) needs, but the periodic identification requires (7). Every later use of "the solid invariant" in this synthesis carries this qualification.

Recollection 3.5 (Crossed-product disorder algebra; Part II, Theorem II-C [28])**.** *For $\Omega = Q^{\mathbb{Z}^d}$ with the shift action T, the covariant observable algebra of a homogeneous disordered family is the crossed product $C(\Omega) \rtimes_T \mathbb{Z}^d$ [7], a separable unital C^*-algebra whose K_0-trace range is the gap-labelling group; its condensation is functorial in the finite quotients of Ω.*

Emergent structure. The composition of Arrows I and II produces *functorial topological invariants of disordered families over profinite hulls*, computed by an intrinsically condensed operation (solidification) rather than imported by hand. This is the first place the condensed language does work an ordinary smooth-parameter treatment does not: the profinite probes of (1) match the physical configuration space Ω of Recollection 3.5, they do not approximate it. Part II's Conjectures II-1 through II-5 record what is open here.

3.3 Arrow III: the uniformly gapped substack

The third arrow selects the systems on which a phase invariant is even defined. The selection is governed by a wall.

Recollection 3.6 (Undecidability of the gap; Cubitt–Pérez-García–Wolf [10])**.** *There is a family of translation-invariant nearest-neighbour Hamiltonians on a two-dimensional lattice, depending computably on a parameter, for which no algorithm decides whether the thermodynamic-limit system is gapped or gapless; the property is Π_1-complete.*

Consequently membership in $\mathfrak{Gap}_{d,G}$ is not, in general, decidable, and Part III does not attempt to decide it. Existence of a thermodynamic gap *defines* the substack; the content is stability.

Recollection 3.7 (Gap continuity and stability; Part III, Theorems III-A and III-B [29]). *At fixed finite volume the n-gap is $2C_\Lambda$-Lipschitz in the interaction norm, so the finite-volume gapped locus is open and the gap is continuous along continuous families (III-A). On a frustration-free / LTQO stratum, the Bravyi–Hastings–Michalakis and Michalakis–Zwolak stability theorems with the Nachtergaele–Sims–Young bulk-gap result [8, 30, 32] give a size-independent perturbation threshold ε_0: for Φ in the stratum and a quasi-local direction V, the segment $\{\, \Phi + tV : 0 \le t < \varepsilon_0 \,\} \subseteq \mathfrak{Gap}_{d,G}$ with gap bounded below by $\gamma/2$. Hence $\mathfrak{Gap}_{d,G}$ is condensed-open along the quasi-local direction class at each stratum point.*

Recollection 3.8 (Uniform clustering; Part III, Theorem III-C [29]). *A uniformly gapped family over a profinite base, with a uniform Lieb–Robinson velocity from Recollection 3.2, has exponential clustering with constants—hence a correlation length—bounded uniformly over the base [16, 31].*

Emergent structure. Inverting $\mathcal{W}$ and stabilizing on $\mathfrak{Gap}_{d,G}$ yields a *robust* π_0: a set of phases stable under quasi-local deformation, disorder averaging, and finite-depth circuits. The robustness is exactly what Recollections 3.7 and 3.8 buy: openness so a phase label does not change under small perturbations, clustering so it is detected locally. But the openness holds only on the stratum where the stability hypotheses hold; Conjecture III-1 asks for it on the full physical stratum, III-2 for descent along hulls, III-3 for the identification of $\mathcal{W}$-components with Ogata's operator-algebraic phases [33].

3.4 Arrow IV: stabilization and the invertible condensed phase spectrum

The fourth arrow group-completes the invertible sector into a spectrum. On components $\boxtimes$ makes $\mathrm{Phases}_{d,G}$ a commutative monoid with unit the trivial product state; the short-range-entangled (SRE) phases are exactly its invertible elements.

Recollection 3.9 (Group completion and the stabilization element; Part IV [26]). *For a commutative monoid M, the Grothendieck group* $\mathrm{Groth}(M)$ *has the universal property that monoid maps $M \to A$ into abelian groups factor uniquely through $M \to \mathrm{Groth}(M)$; $\gamma_M(a) = \gamma_M(b)$ iff $a + e = b + e$ for some $e \in M$, so γ_M is injective iff M is cancellative. The element e is the algebraic shadow of ancilla stabilization.*

Recollection 3.10 (Kubota's Ω-spectrum; Part IV, after Kubota [24, 26]). *There is an Ω-spectrum IP^*, built from the operator-algebraic formulation of invertible gapped quantum spin systems, whose homotopy groups are the groups of invertible gapped systems in each dimension. It realizes Kitaev's proposal [23] that invertible phases are the homotopy groups of a spectrum and is the rigorous carrier for the homotopy of $\mathbf{IP}^{\mathrm{cond}}_{d,G}$.*

Applying the recognition principle for grouplike E_∞-spaces to the invertible sector yields a connective spectrum, the invertible condensed phase spectrum $\mathbf{IP}^{\mathrm{cond}}_{d,G}$ [6]. Against it Part IV sets the effective-field-theory target: the Freed–Hopkins Anderson-dual bordism spectrum I_{FH} classifying invertible topological field theories [15, 14].

Recollection 3.11 (Hall conductance as a phase invariant; Part IV, after Kapustin–Sopenko [19, 20, 26])**.** *For a two-dimensional SRE lattice state the Hall conductance is locally computable, an integer multiple of* e^2/h*, and constant on gapped phases, hence a homomorphism* $\mathrm{Phases}_{2,G}^{\times} \to \mathbb{Z}$*; the higher Berry class generalizes it to families and unifies it with the Thouless pump [40].*

Emergent structure. Two things appear only at the spectrum level. First, the *higher homotopy*: π_1 of the phase object is loops of gapped Hamiltonians, which implement adiabatic pumps and automorphisms of topological order [1]; π_n is n-parameter families and higher defects. Second, the *transition calculus* becomes a long-exact-sequence computation, the relative charge $\partial\nu$ appearing as a boundary map (Section 6). Part IV's Conjectures IV-1 through IV-5 govern the condensed refinement, the lattice–EFT comparison, the boundary-map identity, the compatibility of solidification with group completion, and the renormalization functor.

3.5 Arrow V: realizability

The fifth arrow asks the converse question: which classes in $\mathbf{IP}_{d,G}^{\mathrm{cond}}$ are produced by an explicit uniformly gapped lattice Hamiltonian? Part V formalizes realizability as essential surjectivity on π_0 of the comparison map real from the stack of lattice models to the abstract spectrum, and separates realizability by *some* gapped local model from realizability by a local *commuting-projector* model.

Recollection 3.12 (Cohomological realizability; Part V, Theorem V-A [27])**.** *For every finite* G *and every* $\omega \in H^{d+1}(G, U(1))$ *there is an explicit* G*-symmetric local commuting-projector Hamiltonian* H_ω *with a unique short-range-entangled ground state on any closed* d*-manifold, whose boundary carries the anomalous* G*-action with obstruction* ω*;* $\omega \mapsto [H_\omega]$ *is a homomorphism* $H^{d+1}(G, U(1)) \to \mathrm{SRE}_{d,G}$ *injective on the group-cohomology subgroup [9, 13].*

Recollection 3.13 (Low-dimensional completeness; Part V, Theorem V-B and Remark V-B′ [27])**.** *For on-site finite symmetry in* $d = 1$*, Ogata's* $H^2(G, U(1))$*-valued index is a* complete *invariant of the symmetric phase [34, 33], so* real *is a bijection onto* $H^2(G, U(1))$ *and realized = invariant. In* $d = 2$ *the situation is genuinely weaker: Ogata's* $H^3(G, U(1))$*-valued index* exists *and is a well-defined invariant [35], and together with the group-cohomology models of Recollection 3.12 this makes* real surjective *onto the in-cohomology classes—so every such class is realized—but whether the index is* complete *(separates all* $d = 2$ *SPT phases with on-site finite symmetry) is open. We therefore do* not *assert realized = invariant in* $d = 2$*; only the surjectivity half is settled there.*

Recollection 3.14 (Commuting-projector no-gos; Part V, Theorem V-C [18, 21, 27])**.** *A local commuting-projector Hamiltonian in two dimensions obeys two logically independent no-gos. (i)* Electric *(Kapustin–Fidkowski [18]): if it is* $U(1)$*-symmetric, its zero-temperature electric Hall conductance vanishes,* $\sigma_H = 0$*. (ii)* Thermal *(Kapustin–Spodyneiko [21]): its chiral central charge vanishes,* $c_- = 0$*, with no symmetry hypothesis. Hence a* $U(1)$*-symmetric phase with* $\sigma_H \neq 0$ *(the integer quantum Hall states, Chern insulators) is excluded* *by (i), and* any *phase with $c_- \neq 0$—including the bosonic E_8 state, which carries no $U(1)$ charge (so $\sigma_H = 0$) yet has $c_- = 8$—is excluded by the chiral mechanism (ii), not by σ_H.*

Emergent structure. The composition of all five arrows produces an *obstruction subgroup* $\mathrm{Obs}_{d,G} = \pi_0 \mathbf{IP}^{\mathrm{cond}}_{d,G} / \operatorname{im} \mathrm{real}$ separating abstract classes from lattice-realizable ones, with the chiral classes of Recollection 3.14 sitting inside the commuting-projector obstruction. Part V's Conjectures V-1, V-2, V-3 describe the image, the chiral classes, and profinite-family realizability.

4 Modular composition

The previous section walked the arrows in isolation. Here we make the *composition* explicit: how the modules compose, what emerges from composition that is invisible in any single module, and what bounds the composition from outside.

4.1 The composition diagram

Read as modules, the five Parts form a diagram in which each object is the input to the next and each module contributes a functor. Write Loc, Obs, Gap, Stab, Real for the five modules.

$$\begin{tikzcd}
\{\text{loc. int.}\} \arrow[r, "\mathsf{Loc}"] & \mathfrak{Ham}_{d,G} \arrow[r, "\mathsf{Gap}"] \arrow[d, "\mathsf{Obs}"'] & \mathfrak{Gap}_{d,G} \arrow[r, "\mathsf{Stab}"] & \mathfrak{Phase}_{d,G} \arrow[r, "\boxtimes,\ \mathrm{Groth}"] & \mathbf{IP}^{\mathrm{cond}}_{d,G} \arrow[dl, dashed, "\mathrm{real}"'] \arrow[d, "c_{d,G}"] \\
 & (\mathcal{A},\ \underline{\mathcal{S}(\mathcal{A})},\ \mathrm{Solid}\, K_{\mathrm{alg}}) & & \{\text{realized}\} & I_{\mathrm{FH}}
\end{tikzcd} \tag{8}$$

Here Loc, Obs, Gap, Stab, and the group-completion arrow are the functors of Parts I, II, III, III–IV, and IV; the solid arrows are functors the modules construct, with the caveat that Stab inverts $\mathcal{W}$, whose generation from quasi-adiabatic continuation is Conjecture III-3. The dashed arrow real is the realization comparison of Section 3.5 (Part V), whose essential image is Conjecture V-1; the vertical comparison $c_{d,G} : \mathbf{IP}^{\mathrm{cond}}_{d,G} \to I_{\mathrm{FH}}$ to the Freed–Hopkins target is Conjecture IV-2. The observable data $(\mathcal{A}, \underline{\mathcal{S}(\mathcal{A})}, \mathrm{Solid}\, K_{\mathrm{alg}})$ produced by Obs feeds the homotopy of the spectrum through the solid refinement of Conjecture IV-1. The diagram commutes at the level of π_0 exactly where those conjectures hold—in $d = 1$ for on-site finite symmetry (Recollection 3.13), and for the free-fermion tenfold way [23, 2]—and its global commutativity is the Master Conjecture of Section 9.

4.2 Emergent properties at each level

The characteristic feature of a modular composition is that structure emerges at each level which is not a property of the parts. We collect the four emergences named in Section 3.

1. **Uniformity** (I$\circ$nothing). A single light cone, clustering length, and constant set over a whole profinite base: the precondition for every "family" below.

2. **Functorial disordered invariants** (II∘I). Topological invariants of disordered families computed by solidification, matched to the physical configuration space rather than a smooth approximation of it.

3. **A robust phase set** (III∘II∘I). $\mathrm{Phases}_{d,G} = \pi_0\, \mathrm{Shape}\, \mathfrak{Phase}_{d,G}$, stable under quasi-local deformation once $\mathcal{W}$ is inverted and ancillas added.

4. **Higher homotopy and an obstruction subgroup** (V∘IV∘III∘II∘I). Pumps and defects as $\pi_{\geq 1}$, and $\mathrm{Obs}_{d,G}$ separating abstract from realizable.

None of these emergences is forced by the module below it alone; each requires the composite. That is the precise content of calling the program modular rather than layered: the modules interoperate, and the interoperation is where the physics of families, disorder, and defects lives.

4.3 The two walls

Two theorems bound the composition and must be respected at every level.

The *undecidability wall* (Recollection 3.6) forbids any claim that $\mathfrak{Gap}_{d,G}$ is decidable or algorithmically presentable. It does not forbid stability results: openness on a stratum, uniform clustering, and descent along a hull are all statements about the *neighborhood* of a gapped system, not tests for gappedness, and Part III is careful to scope them to strata with uniform constants. The wall is why the program is organized around stability, not existence.

The *no-go wall* (Recollection 3.14), combining the electric Kapustin–Fidkowski and the thermal Kapustin–Spodyneiko no-gos, forbids any claim that a chiral invertible class is realized by commuting projectors. It bounds $\mathrm{im}\,\mathrm{real}^{\mathrm{cp}}$ to the non-chiral sector and makes the separation between "exactly solvable" and "physically realizable" structural, not incidental. Every realizability statement in the program excludes the chiral commuting-projector case by hypothesis.

A third boundary is not a theorem but a scope limit: a spectrum classifies only invertible order. General anyon theories and non-invertible phases need higher categories of excitations (braided or modular tensor categories in 2+1 dimensions and their higher analogues), so a condensed treatment of them would require a condensed higher stack of phases and defects, not a condensed K-theory spectrum. That extension is outside the present program.

5 The homotopy dictionary

Classification by π_0 records only which phases exist. The higher homotopy of the phase object carries more, and the program fixes a dictionary between homotopy-theoretic and physical data. We quote it verbatim from the prospectus, now with each line anchored to the module that supplies it.

$$\boxed{\begin{aligned}\pi_0 &= \text{phases},\\ \pi_1 &= \text{adiabatic pumps and phase automorphisms},\\ \pi_n &= \text{higher families and higher defects},\\ \Sigma &= \text{gapless transition locus},\\ E^{q+1}(B, B\setminus\Sigma) &= \text{relative charge of a transition}.\end{aligned}} \tag{9}$$

The first line is Arrow III's emergent structure, made precise by Conjecture III-3: the components of $\mathfrak{Gap}_{d,G}[\mathcal{W}^{-1}]$ are the operator-algebraic gapped phases of Ogata [33]. The second and third are Arrow IV's: loops and higher families of gapped Hamiltonians realize pumps and defects, carried by the homotopy of Kubota's spectrum [24] and, conjecturally, by the higher homotopy solid modules of $\mathbf{IP}^{\mathrm{cond}}_{d,G}$ (Conjecture IV-1); the automorphism content of π_1 is the subject of [1], whose π_1, π_2, π_3 computations are themselves conjectural. The last two lines are the transition calculus, developed next.

6 Phase transitions and the relative-charge formalism

This section develops the fourth and fifth lines of (9): what a transition is, and what its charge is. The development is honest obstruction theory (the long exact sequence of a pair and excision, valid for any generalized cohomology), and the only conjectural step is the identification of the abstract boundary map with the spectral boundary map of $\mathbf{IP}^{\mathrm{cond}}_{d,G}$, which is Conjecture IV-3 lifted to the program.

6.1 The gapped locus, the discriminant, and the phase label

Let B be a parameter space (couplings, fields, pressures, disorder configurations, boundary conditions), regarded as a condensed object $\underline{B}$, and let $f : \underline{B} \to \mathfrak{Ham}_{d,G}$ be a family of systems.

Definition 6.1 (gapped locus and discriminant)**.** The *gapped locus* of f is the pullback

$$U_f = \underline{B} \times_{\mathfrak{Ham}_{d,G}} \mathfrak{Gap}_{d,G},$$

and the *gapless (critical) locus* or *discriminant* is its complement $\Sigma_f = \underline{B} \setminus U_f$. On U_f the phase label

$$\nu_f : U_f \longrightarrow \pi_0(\mathfrak{Phase}_{d,G}) = \mathrm{Phases}_{d,G}$$

is locally constant.

That ν_f is locally constant on U_f is exactly Recollection 3.7: the gapped locus is condensed-open on the physical stratum, and openness of each phase's preimage is what "locally constant" means. A topological phase transition is a path in B whose endpoints carry different values of ν_f; such a path cannot stay in U_f and must meet Σ_f. This is the boxed slogan (5).

6.2 The relative charge as an obstruction

Suppose the invariant is valued in a (condensed) generalized cohomology theory E; in the invertible case E is represented by $\mathbf{IP}^{\mathrm{cond}}_{d,G}$. On the gapped locus the invariant is a class $\nu \in E^q(U_f)$. The question of the transition is whether ν extends over the critical set to all of B. Obstruction theory answers it through the long exact sequence of the pair (B, U_f):

$$\cdots \to E^q(B) \xrightarrow{j^*} E^q(U_f) \xrightarrow{\partial} E^{q+1}(B, U_f) \to E^{q+1}(B) \to \cdots, \tag{10}$$

with $j : U_f \hookrightarrow B$ the inclusion.

Definition 6.2 (relative transition charge)**.** The *relative transition charge* of the family f carrying the invariant $\nu \in E^q(U_f)$ is

$$\partial\nu := \partial\nu \in E^{q+1}(B, U_f) = E^{q+1}(B, B \setminus \Sigma_f),$$

the image of ν under the connecting map of (10).

Proposition 6.3 (extension obstruction)**.** *$\partial\nu = 0$ if and only if ν is the restriction of a class on all of B; that is, $\partial\nu$ is the obstruction to extending the phase invariant across the discriminant. In particular, if $\partial\nu \neq 0$ the family has a genuine transition: no global invariant restricts to ν on U_f.*

Proof. Exactness of (10) at $E^q(U_f)$: $\nu \in \mathrm{im}(j^*)$ if and only if $\partial\nu = 0$. A class in $\mathrm{im}(j^*)$ is by definition the restriction of a class on B. This is the standard long exact sequence of a pair in a generalized cohomology theory and requires nothing beyond the Eilenberg–Steenrod axioms that E satisfies as a spectrum-represented theory. □

6.3 Locality: linking spheres

The relative group $E^{q+1}(B, B \setminus \Sigma_f)$ is local along Σ_f. When Σ_f is a reasonable (say, closed, locally-flat, finite-codimension) condensed subobject with components $\{\Sigma_f^{(\alpha)}\}$, excision identifies the relative cohomology with a sum of contributions supported near the components:

$$E^{q+1}(B, B \setminus \Sigma_f) \;\cong\; \bigoplus_\alpha E^{q+1}\big(N_\alpha, N_\alpha \setminus \Sigma_f^{(\alpha)}\big), \tag{11}$$

N_α a tubular neighborhood of $\Sigma_f^{(\alpha)}$. If $\Sigma_f^{(\alpha)}$ has codimension c with an E-oriented normal bundle, the Thom isomorphism identifies each summand with a degree-shifted absolute group of the component,

$$E^{q+1}\big(N_\alpha, N_\alpha \setminus \Sigma_f^{(\alpha)}\big) \;\cong\; E^{q+1-c}\big(\Sigma_f^{(\alpha)}\big), \tag{12}$$

the Thom class shifting degree by the codimension c. The local charge is read off on a linking sphere: over a point $x \in \Sigma_f^{(\alpha)}$ the unit sphere S^{c-1}_{link} of the normal fibre lies in $B \setminus \Sigma_f$ and links the component, and the restriction $\nu|_{S^{c-1}_{\mathrm{link}}} \in E^q\big(S^{c-1}_{\mathrm{link}}\big)$ has reduced part $\widetilde{E}^q(S^{c-1}) \cong E^{q-c+1}(\mathrm{pt})$ equal to that local charge: the same degree shift as (12). In words: the relative charge decomposes as a sum of local charges, one per component of the critical set, each detected by evaluating E on a small sphere linking the component. This is the precise form of the physical picture that a band degeneracy (a Dirac node, a Weyl point) acts as a source or sink of topological charge, now valid for a generalized cohomology theory and an interacting phase spectrum rather than only for Chern numbers of free bands. This relative, defect-localized reading has a free-fermion precedent: Teo and Kane classify topological defects by the K-theory of a sphere linking the defect [38], within the K-theoretic classification of topological phases [39]; the condensed formulation of Definition 6.2 is the extension of that relative-charge picture to a generalized theory represented by $\mathbf{IP}^{\mathrm{cond}}_{d,G}$. The low-codimension case $c = 1$ (a wall Σ_f separating two gapped regions) reduces the linking sphere to S^0: two points, one in each phase, and the local charge is the *difference* of the two phase labels: the jump of ν_f across the wall.

6.4 The spectral boundary map

Proposition 6.3 and (12) are unconditional facts about any E. The program's substantive claim is that, for $E = \mathbf{IP}^{\mathrm{cond}}_{d,G}$-cohomology, the connecting map ∂ of (10) is the boundary map of the cofiber sequence of the pair for the spectrum $\mathbf{IP}^{\mathrm{cond}}_{d,G}$, so the transition charge is computed by a single long exact sequence in every dimension and every generalized theory simultaneously. This is Conjecture IV-3, restated at the program level as Conjecture VI-3 in Section 9. Its low-dimensional shadow (the SSH winding jump, the Dirac node as a unit source of charge) is classical; the conjecture is that this is systematically the spectral boundary map.

7 The SSH transition end to end

We now run one example through all five modules, to show that the composition is not merely formal. The Su–Schrieffer–Heeger chain [37] is the smallest system in which every module has something to say, and its winding number is a degree-≥ 0 class, so it sits safely on the nonnegative side of the Bott-inversion caveat of Recollection 3.4.

Example 7.1 (SSH through the spine)**.** The SSH Bloch Hamiltonian and its off-diagonal function are

$$H(k; t_1, t_2) = (t_1 + t_2 \cos k)\, \sigma_x + (t_2 \sin k)\, \sigma_y, \qquad q(k) = t_1 + t_2 e^{ik},$$

with spectrum $E_\pm(k) = \pm|q(k)|$.

Proposition 7.2 (gap and winding of the SSH chain)**.** *The chain is gapped if and only if $|t_1| \neq |t_2|$; the gap closes on the discriminant $\Sigma = \{\, (t_1, t_2) : |t_1| = |t_2| \,\}$. On the gapped locus the winding number of $q : S^1 \to \mathbb{C}^\times$ is*

$$\nu = \begin{cases} 1, & |t_1| < |t_2|, \\ 0, & |t_1| > |t_2|, \end{cases}$$

up to orientation, and jumps by ± 1 across Σ.

Proof. $E_\pm(k) = \pm|q(k)|$ vanishes for some k iff $q(k) = 0$, i.e. iff $t_1 + t_2 e^{ik} = 0$ for some k, which happens iff $|t_1| = |t_2|$; hence the gap is open exactly off Σ. On the gapped locus q misses the origin, so its winding number is defined; for $|t_1| > |t_2|$ the curve $k \mapsto t_1 + t_2 e^{ik}$ is a circle of radius $|t_2|$ about $t_1 \neq 0$ not enclosing the origin (winding 0), and for $|t_1| < |t_2|$ it encloses the origin once (winding 1). The two regions are separated by Σ, across which the winding jumps by ± 1. $\square$

Now the five modules, in order.

Part I (dynamics). The SSH couplings define a finite-range interaction of finite F-norm, so the family lives in $\underline{\mathcal{B}_F}$ and, by Recollection 3.2, over any compact region of the (t_1, t_2)-plane away from Σ carries one Lieb–Robinson velocity. The parameter torus and Brillouin circle are replaced by their condensations $\underline{B}, \underline{S^1}$; the light cone is uniform over the probe.

Part II (algebra and K-theory). SSH is a class $AIII$ system in $d = 1$, whose tenfold-way entry is $\mathbb{Z}$ [23, 2], the winding number valued in K^1 of the observable algebra. Because the winding is a nonnegative-degree class, Recollection 3.4 applies without invoking the Bott inversion: the solid invariant $\mathrm{Solid}(K_{\mathrm{alg}}(\underline{A}))$ computes it directly, and its provenance is condensed even though the number is the classical winding.

Part III (gap). Proposition 7.2 is exactly the finite-volume/thermodynamic gap statement of Recollection 3.7 for this family: the gapped locus is $U = \{\, |t_1| \neq |t_2| \,\}$, open, and the phase label ν is locally constant on it, with the discriminant $\Sigma = \{\, |t_1| = |t_2| \,\}$.

Part IV (stabilization and EFT). The low-energy theory at the gap-closing $|t_1| = |t_2|$, $k = \pi$ is a massive 1+1-dimensional Dirac fermion, and the winding number is the sign of the Dirac mass—the deformation class of the Dirac effective field theory. So the comparison map $c_{1,AIII}$ sends the SSH phase to its EFT class and the two agree: Conjecture IV-2 holds at the level of π_0 in this box.

Part V (realizability). The nontrivial SSH phase is short-range-entangled and realized by the explicit dimerized chain; being non-chiral ($\sigma_H = 0$ and $c_- = 0$), it is excluded by neither no-go of Recollection 3.14 and is in fact commuting-projector realizable in the flat-band (fully dimerized) limit. It sits in $\mathrm{im}\,\mathrm{real}$.

The transition charge. In the parameter plane $B = (t_1, t_2)$ the invariant relevant to the transition calculus is not the momentum-space K^1 class of the tenfold way but the *winding number* it produces: a locally constant $\mathbb{Z}$-valued label on the gapped locus, an element $\nu \in K^0(U)$ of degree $q = 0$. The discriminant $\Sigma = \{\, |t_1| = |t_2| \,\}$ is the pair of lines $t_1 = \pm t_2$; away from their intersection it has codimension $c = 1$, so the linking sphere of (12) is S^0 and the local charge is read in the reduced group $\widetilde{K}^0(S^0) \cong \mathbb{Z}$—the winding difference ± 1 across the wall, in agreement with the degree shift $E^{q-c+1}(\mathrm{pt}) = K^0(\mathrm{pt}) = \mathbb{Z}$ of (12). (At the origin $(0,0)$ the two lines cross, the gap closes at $k = 0$ and $k = \pi$ simultaneously, and the codimension jumps to 2; the generic crossing is the codimension-one one just described.) By Proposition 6.3 the relative charge $\partial\nu \in K^1(B, U)$ is nonzero precisely because the winding label does not extend across Σ: the transition is real, and its charge is the unit jump. Conjecture VI-3 is the assertion that this ∂ is the spectral boundary map of $\mathbf{IP}^{\mathrm{cond}}_{1,AIII}$; here it is the elementary winding jump, and the two agree.

8 Disorder and the profinite thread

The profinite disorder thread is where condensed mathematics does work that ordinary smooth-parameter topology cannot, and it runs through all five modules. We collect it.

8.1 The hull as a profinite probe

If Q is a finite set of local configurations, the configuration space

$$\Omega = Q^{\mathbb{Z}^d} \tag{13}$$

is compact, totally disconnected, hence profinite; this is the disorder (or tiling) hull of Bellissard's noncommutative geometry of aperiodic media [7, 22]. (We follow Part II in writing the alphabet Q rather than the prospectus's F, to avoid collision with the Nachtergaele–Sims–Young F-function of Part I; the two never denote the same object.) A disordered Hamiltonian family $\omega \mapsto H_\omega$ is literally an Ω-point of the moduli stack, $H \in \mathfrak{Ham}_{d,G}(\Omega)$, and finite quotients of Ω are finite-resolution disorder data. The sheaf condition of (1) is the compatibility of families and invariants across finite approximations, so the profinite probes match the physical configuration space rather than repackage a smooth one.

8.2 The thread through the modules

- **Part I** (Conjecture I-3): the Lieb–Robinson estimates over Ω should be the right Kan extension of their restrictions to finite quotients: quasi-local dynamics determined by finite-resolution data. Its unconditional shadow is the uniform light cone of Recollection 3.2.
- **Part II** (Recollection 3.5, Conjecture II-2): the observable algebra is the crossed product $C(\Omega) \rtimes_T \mathbb{Z}^d$, its condensation functorial in finite quotients, and the solid invariant should agree with operator K-theory naturally in Ω.
- **Part III** (Conjecture III-2): uniform gappedness over Ω should be a closed condition on the inverse system: the uniformly gapped families the inverse limit of the finite-resolution ones at a fixed Δ.
- **Part IV**: parametrized invariants over Ω, the base level (L2) of the lattice–EFT comparison Conjecture IV-2.
- **Part V** (Conjecture V-3): realizability over Ω is disorder-robust iff a condensed-cohomological descent obstruction $o(c) \in H^1_{\mathrm{cond}}(\Omega, \underline{\mathrm{Aut}}(c))$ vanishes, detected by finite quotients.

The common shape of all five is descent along the finite quotients of Ω. That the five descent statements are facets of one principle is Conjecture VI-4.

9 The program-level conjectures

We now state the open claims of the program. Each is built as the coherent join of module-level conjectures from Parts I–V, and none is proved. We write them as named Conjectures VI-1 through VI-4; the first is the Master Conjecture.

Conjecture VI-1 (Master Conjecture: the modules glue into one condensed higher stack)**.** *The five module-level structural conjectures hold simultaneously and coherently:* $\mathfrak{Ham}_{d,G}$ *is a condensed higher stack (Conjecture I-1); positivity and the* C^**-identity cut it out as a closed condensed substack of a formal-interaction stack (Conjecture II-1);* $\mathfrak{Gap}_{d,G}$ *is an open condensed substack on the physical stratum with the uniform-gap sheaf condition detected by finite quotients (Conjecture III-1); the stabilized invertible sector is a connective condensed/solid spectrum* $\mathbf{IP}^{\mathrm{cond}}_{d,G}$ *whose shape is a connective cover of Kubota's* IP^* *(Conjecture IV-1); and the realizability image is the short-range-entangled subspectrum (Conjecture V-1). Consequently the boxed sequence* (6) *is a diagram of condensed higher stacks and spectra,* $\mathrm{Phases}_{d,G} = \pi_0\,\mathrm{Shape}(\mathfrak{Phase}_{d,G})$ *is the operator-algebraic set of gapped phases, and* $\mathbf{IP}^{\mathrm{cond}}_{d,G}$ *is the invertible phase spectrum of the whole.*

Conjecture VI-2 (microscopic = field-theoretic)**.** *Under short-range-entanglement hypotheses the comparison map* $c_{d,G} : \mathbf{IP}^{\mathrm{cond}}_{d,G} \to I_{\mathrm{FH}}$ *to the Freed–Hopkins invertible-TQFT spectrum is an equivalence after solidification/completion (Conjecture IV-2), and its* π_0 *image coincides with the realized SRE subspectrum* $\mathrm{im\,real} = \mathrm{SRE}_{d,G}$ *(Conjecture V-1). Thus the microscopic classification of* $\mathrm{Phases}^{\times}_{d,G}$ *and the bordism classification of* $\pi_0 I_{\mathrm{FH}}$ *agree exactly on the realizable classes, with the chiral part realizable by non-commuting gapped models but never by commuting projectors (Conjecture V-2, Recollection 3.14). The statement holds at* π_0 *in* $d = 1$ *for on-site finite symmetry (Recollection* 3.13*) and for the free-fermion tenfold way, and is open in general.*

Conjecture VI-3 (the relative charge is a spectral boundary map)**.** *For* $E = \mathbf{IP}^{\mathrm{cond}}_{d,G}$*-cohomology, the connecting map* ∂ *of* (10) *is the boundary map of the cofiber sequence of the pair* (B, U_f) *for the spectrum* $\mathbf{IP}^{\mathrm{cond}}_{d,G}$ *(Conjecture IV-3). Hence the relative transition charge* $\partial\nu = \partial\nu \in E^{q+1}(B, B \setminus \Sigma_f)$ *of Definition 6.2 is computed by a single long exact sequence in every dimension and generalized theory, and its linking-sphere localization* (12) *is the systematic form of "a band degeneracy is a source of topological charge." The SSH box (Example 7.1) is the codimension-one, winding-number instance.*

Conjecture VI-4 (all program invariants descend along disorder hulls)**.** *For a profinite disorder hull* $\Omega = Q^{\mathbb{Z}^d} = \varprojlim_i \Omega_i$*, the disorder facets of the five modules—Lieb–Robinson descent (I-3), naturality of the solid invariant (II-2), uniform-gap descent (III-2), parametrized comparison over* Ω *(IV-2 at level L2), and profinite-family realizability (V-3)—are facets of one descent principle: every program invariant over* Ω *is the right Kan extension of its restrictions to the finite quotients* Ω_i*, and uniform gappedness, the solid invariant, and realizability are all detected by finite quotients with a common bound. This is the precise sense in which "profinite probes match the configuration space."*

9.1 Consolidated open problems

Table 1 lists the nineteen module-level conjectures of Parts I–V, with the module each depends on and the program-level conjecture it feeds. The dependency column records which *earlier* module's output a conjecture presupposes; the "feeds" column records which Conjecture VI-k consolidates it. The table is the program's to-do list.

Two features of Table 1 are worth naming. First, the dependency graph is a tree rooted at I-1: every conjecture presupposes the stack structure of the ground floor, which is why Conjecture I-1 is the true foundation and the Master Conjecture VI-1 leads with it. Second, the "feeds" column shows the program-level conjectures are not independent wishes but bundles: VI-1 gathers the ten structural conjectures, VI-2 the four comparison/refinement conjectures, VI-3 the single boundary-map conjecture, and VI-4 the four descent conjectures ($10+4+1+4=19$). Proving any program-level conjecture means proving its whole bundle.

Table 1: The nineteen module-level conjectures of Parts I–V and their dependencies. "Feeds" names the program-level Conjecture VI-k that consolidates each.

ID	Statement (abbreviated)	Depends on	Feeds
I-1	$\mathfrak{Ham}_{d,G}$ is a condensed higher stack (descent of the automorphism groupoid)	—	VI-1
I-2	Quasi-local automorphisms form a condensed group; QAC is internal path-lifting	I-1	VI-1
I-3	Lieb–Robinson estimates descend along a disorder hull	I-1	VI-4
II-1	Positivity + C^*-identity cut $\mathfrak{Ham}_{d,G}$ out as a closed condensed substack	I-1	VI-1
II-2	Solid invariant = operator K of the crossed product, naturally in Ω	II-1	VI-4
II-3	Real/KKO refinement recovering the KO-graded periodic table	II-1	VI-2
II-4	Solid state space is a classifying object for condensed representations	II-1	VI-1
II-5	Condensed enhancement of the split property / DHR superselection	II-1	VI-1
III-1	$\mathfrak{Gap}_{d,G}$ open on the physical stratum; sheaf condition detected by finite quotients	II-1	VI-1
III-2	Uniform-gap descent along profinite hulls (closed on the inverse system)	III-1	VI-4
III-3	QAC path components realize $\mathcal{W}$; $\pi_0\,\mathrm{Shape}(\mathfrak{Gap}[\mathcal{W}^{-1}])$ = Ogata phases	III-1	VI-1
IV-1	$\mathbf{IP}^{\mathrm{cond}}_{d,G}$ is a connective condensed/solid spectrum covering Kubota's IP^*	III-3	VI-1
IV-2	Lattice–EFT comparison $c_{d,G} : \mathbf{IP}^{\mathrm{cond}}_{d,G} \to I_{\mathrm{FH}}$ is an equivalence after completion	IV-1	VI-2
IV-3	Relative charge $\partial\nu$ is the spectral boundary map of $\mathbf{IP}^{\mathrm{cond}}_{d,G}$	IV-1	VI-3
IV-4	Solidification commutes with group completion	IV-1	VI-1
IV-5	Renormalization is a filtered pro-endofunctor computing the EFT on the SRE stratum	IV-1	VI-2
V-1	Realizability image = SRE subspectrum; $\mathrm{Obs}_{d,G} = \pi_0\mathbf{IP}^{\mathrm{cond}}_{d,G} / \operatorname{im} \mathrm{real}$	IV-2	VI-1
V-2	Chiral classes realizable by non-commuting gapped, never commuting-projector	V-1	VI-2
V-3	Profinite-family realizability iff a condensed descent obstruction vanishes	V-1	VI-4

10 Contributions and limitations

10.1 Contributions

The framework contributes three structural advantages, none of which is a new numerical invariant. First, one category holds ordinary continuous families, profinite disorder families, compact inverse limits, and topological symmetry groups at once; the passage (1) is fully faithful, so nothing classical is lost. Second, condensed abelian groups and solid modules supply exact derived operations and descent where ordinary topological groups and modules are poorly behaved: the environment Recollection 3.4 needs to make operator K-theory a condensed invariant. Third, the sheaf and stack viewpoint organizes local families, defects, interfaces, boundary conditions, symmetry actions, and gluing uniformly, which is what the transition calculus of Section 6 and the disorder thread of Section 8 exploit.

On novelty we are precise, because the reviewer should be. Condensed and pyknotic mathematics [36, 5] and the solidification bridge [3] are established in pure mathematics; the moduli/space-of-states viewpoint on gapped systems is established in physics [17, 6, 24, 11, 12]. What is new is the *synthesis*: no prior work applies condensed or pyknotic machinery to topological phases, and the individual analytic ingredients (Lieb–Robinson bounds, C^*-positivity, gap stability, stabilization, realizability) are not new in themselves. The value is unificatory and organizational, and the honest content is the Master Conjecture and its bundle.

10.2 What it does not do

By itself the paradigm does not solve the hard analysis of many-body theory: one still needs locality estimates (Part I), positivity and C^*-norm conditions (Part II), the existence and stability of a thermodynamic gap (Part III), the lattice–EFT correspondence (Part IV), and physical realizability (Part V), and each is a genuine theorem or a genuine conjecture, not a corollary of the formalism. It does not decide the gap: the undecidability wall (Recollection 3.6) is permanent. It does not realize chiral phases by commuting projectors: the no-go wall (Recollection 3.14) is permanent. It does not, by construction, classify non-invertible topological order, which needs a condensed higher stack of phases and defects rather than a spectrum. And it does not produce a different Chern number: the winding of Proposition 7.2 is the classical winding; only its provenance is condensed. The genuinely new thesis is the single environment, and the genuinely open mathematics is Conjectures VI-1 through VI-4.

11 Conclusion

The six papers of this series are one program with five modules and a spectrum-level payoff. Part I makes the interaction space condensed and the dynamics a base-uniform morphism; Part II attaches observable algebras, a compact condensed state space, and the solid K-invariant with its Bott-inversion bookkeeping; Part III cuts out the uniformly gapped substack and stabilizes it against the undecidability wall; Part IV group-completes the invertible sector into $\mathbf{IP}^{\mathrm{cond}}_{d,G}$ and confronts the field-theoretic classification; Part V asks which classes a lattice realizes, bounded by the commuting-projector no-gos of Kapustin–Fidkowski and Kapustin–Spodyneiko. Composed, the modules give the boxed sequence (6), the homotopy dictionary (9), and the transition calculus in which a transition is the crossing of the discriminant and its charge is the relative class $\partial\nu$ localized by linking spheres.

We have been careful to keep the ledger honest. The module results are theorems, cited to Parts I–V. The program results (that the modules glue into one condensed higher stack with $\mathbf{IP}^{\mathrm{cond}}_{d,G}$ its invertible spectrum, that microscopic and field-theoretic classifications agree, that the relative charge is a spectral boundary map, that all invariants descend along disorder hulls) are Conjectures VI-1 through VI-4, and Table 1 records exactly what each would need. The program is not a completed classification and does not claim to be. It is a single categorical and homological environment in which continuous families, profinite disorder, analytic completions, operator K-theory, symmetry, stacking, defects, and phase-transition loci can be treated at once, and a precise list of what remains to prove.

Code availability

The formal-verification suites accompanying the six-paper series are at github.com/YonedaAI/topological-phases-of-matter; each module's code lives in a per-topic subdirectory (for example `src/bordism-realizability/` and `src/lattice-eft-equivalence/`). This synthesis introduces no new code of its own; it composes the results verified in Parts I–V.

Acknowledgements

This is Part VI of a six-part series by the author and The YonedaAI Collaboration; it depends entirely on the analytic and topological content of Parts I–V and on the foundational work cited throughout.

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